Symmetries and stabilization for sheaves of vanishing cycles

Chris Brav, Vittoria Bussi, Delphine Dupont,
Dominic Joyce, and Balázs Szendrői

Abstract

We study symmetries and stabilization properties of perverse sheaves of vanishing cycles $\mathcal{PV}^\bullet_{V,f}$ of holomorphic functions $f:V\to\mathbb{C}$ on complex manifolds. We first prove that if $f:V\to\mathbb{C}$ is holomorphic with critical locus $X=\mathrm{Crit}(f)$, $\Phi:V\to V$ is a local biholomorphism near X, fixing X and compatible with f, then the action of Φ_* on $\mathcal{PV}^\bullet_{V,f}$ is multiplication by $\det\bigl(\mathrm{d}\Phi|_X\bigr)=\pm 1$. We then consider three different situations in which $f:V\to\mathbb{C}$, $g:W\to\mathbb{C}$ are holomorphic on complex manifolds V,W with $\mathrm{Crit}(f)\cong\mathrm{Crit}(g)$ satisfying extra compatibility conditions. We construct canonical isomorphisms $\mathcal{PV}^\bullet_{V,f}\to\mathcal{PV}^\bullet_{W,g}\otimes_{\mathbb{Z}_2}P_{f,g}$, where $P_{f,g}$ is a natural principal $\mathbb{Z}_2$-bundle on the common critical locus. We also generalize our results to mixed Hodge modules of vanishing cycles.

These results will be used in the sequels [6, 7] to construct perverse sheaves and mixed Hodge modules on moduli schemes of stable coherent sheaves on Calabi–Yau 3-folds over $\mathbb{C}$ equipped with orientation data, giving a categorification of Donaldson–Thomas invariants, and to categorify intersections of Lagrangians in a complex symplectic manifold.

Contents

1 Introduction

Let V be a complex manifold and $f : V \to \mathbb{C}$ be holomorphic, and write $X = \mathrm{Crit}(f)$, as a complex analytic subspace of V. Then one can define the *perverse sheaf of vanishing cycles* $\mathcal{PV}^\bullet_{V,f}$ on X. Formally, $X = \coprod_{c \in f(X)} X_c$, where $X_c \subseteq X$ is the open and closed complex analytic subspace of points $x \in X$ with $f(x) = c$, and $\mathcal{PV}^\bullet_{V,f}|_{X_c} = \phi^p_{f-c}(\mathbb{Q}_V[\dim V])|_{X_c}$ for each $c \in f(X)$, where $\mathbb{Q}_V[\dim V]$ is the constant perverse sheaf on V, and $\phi^p_{f-c} : \mathrm{Perv}(V) \to \mathrm{Perv}(f^{-1}(c))$ is the vanishing cycle functor for $f - c : V \to \mathbb{C}$. See §2 for an introduction to perverse sheaves, and an explanation of this notation.

We will consider the following questions:

Question 1.1. (a) *Let V be a complex manifold, $f : V \to \mathbb{C}$ a holomorphic function, and $X = \mathrm{Crit}(f)$. Suppose $\Phi : V \to V$ is a biholomorphism defined near X with $f \circ \Phi = f$ and $\Phi|_X = \mathrm{id}_X$. Then as in Definition* 2.17 *we can define a natural isomorphism $\Phi_* : \mathcal{PV}^\bullet_{V,f} \to \mathcal{PV}^\bullet_{V,f}$, the action of the symmetry Φ of (V, f) on the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{V,f}$.*

What can we say about Φ_*, for instance, when is Φ_* is the identity?

(b) *For V, f, X as above, to what extent is $\mathcal{PV}^\bullet_{V,f}$ invariant under changes to f away from X?*

(c) *Let W be a complex manifold, $V \subseteq W$ a complex submanifold, and $g : W \to \mathbb{C}$ be holomorphic, so that $f := g|_V : V \to \mathbb{C}$ is also holomorphic. Write $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$, as complex analytic subspaces of V, W, and suppose $X = Y$. Then $\mathcal{PV}^\bullet_{V,f}$ and $\mathcal{PV}^\bullet_{W,g}$ are both perverse sheaves on X. What is the relation between them?*

(d) *Let X be a complex analytic space, V,W be complex manifolds, $f:V\to\mathbb{C}$, $g:W\to\mathbb{C}$ be holomorphic functions, and $j:X\to\mathrm{Crit}(f)$, $k:X\to\mathrm{Crit}(g)$ be isomorphisms of complex analytic spaces. Then $j^*(\mathcal{PV}_{V,f}^\bullet)$ and $k^*(\mathcal{PV}_{W,g}^\bullet)$ are both perverse sheaves on X. What is the relation between them?*

Our main results, Theorems 3.1, 4.1, 5.1–5.2 and 6.1, 6.4, 6.7 respectively, present our answers to Question 1.1(a)–(d). In brief, they say:

(a) In Question 1.1(a), $\mathrm{d}\Phi|_{TV|_X}:TV|_X\to TV|_X$ is an automorphism of the vector bundle $TV|_X$ on X, so it has a determinant $\det\bigl(\mathrm{d}\Phi|_X\bigr):X\to\mathbb{C}\setminus\{0\}$. In fact $\det\bigl(\mathrm{d}\Phi|_X\bigr)$ maps to $\{\pm1\}$ in $\mathbb{C}\setminus\{0\}$, and $\Phi_*:\mathcal{PV}_{V,f}^\bullet\to\mathcal{PV}_{V,f}^\bullet$ is multiplication by $\det\bigl(\mathrm{d}\Phi|_X\bigr)$.

(b) Let V be a complex manifold, $f:V\to\mathbb{C}$ be holomorphic, and $X=\mathrm{Crit}(f)$. Write I_X for the ideal of holomorphic functions on V vanishing on X. Suppose $g:V\to\mathbb{C}$ is holomorphic with $f+I_X^3=g+I_X^3$. Then $\mathrm{Crit}(g)=X$ near X, so making V smaller we can suppose $\mathrm{Crit}(g)=X$. There is a natural isomorphism
$$\Lambda_{f,g}:\mathcal{PV}_{V,f}^\bullet\longrightarrow\mathcal{PV}_{V,g}^\bullet.$$
These $\Lambda_{f,g}$ are functorial, i.e. $\Lambda_{f,h}=\Lambda_{g,h}\circ\Lambda_{f,g}$, $\Lambda_{g,f}=\Lambda_{f,g}^{-1}$, $\Lambda_{f,f}=\mathrm{id}$.

(c) In Question 1.1(c), locally there exist biholomorphisms $W\cong V\times\mathbb{C}^n$ identifying V with $V\times\{0\}$ and $g:W\to\mathbb{C}$ with $f\boxplus z_1^2+\cdots+z_n^2:V\times\mathbb{C}^n\to\mathbb{C}$. So locally we have isomorphisms of perverse sheaves
$$\mathcal{PV}_{V,f}^\bullet\cong\mathcal{PV}_{V,f}^\bullet\overset{L}{\boxtimes}\mathcal{PV}_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}^\bullet\cong\mathcal{PV}_{V\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}^\bullet\cong\mathcal{PV}_{W,g}^\bullet,\tag{1.1}$$
using $\mathcal{PV}_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}^\bullet\cong\mathbb{Q}_{\{0\}}$ in the first step, and the Thom–Sebastiani Theorem for perverse sheaves in the second.

This local isomorphism $\mathcal{PV}_{V,f}^\bullet\cong\mathcal{PV}_{W,g}^\bullet$ in (1.1) is only natural up to sign, as $\mathcal{PV}_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}^\bullet\cong\mathbb{Q}_{\{0\}}$ is only natural up to sign, where the sign depends on an orientation for the complex Euclidean space $(\mathbb{C}^n,\mathrm{d}z_1^2+\cdots+\mathrm{d}z_n^2)$. Globally, there is a natural isomorphism
$$\Theta_{f,g}:\mathcal{PV}_{V,f}^\bullet\longrightarrow\mathcal{PV}_{W,g}^\bullet\otimes_{\mathbb{Z}_2}P_{f,g},$$
where $P_{f,g}\to X$ is a principal $\mathbb{Z}_2$-bundle parametrizing choices of orientation of a nondegenerate holomorphic quadratic form q on $\nu|_X$, with ν the normal bundle of V in W, and roughly $q=\partial^2g|_{\nu|_X}$.

These $\Theta_{f,g}$ are functorial, in that if $U\subseteq V$ is a complex submanifold and $e=f|_U$ has $\mathrm{Crit}(e)=X$, then there is a natural isomorphism of $\mathbb{Z}_2$-bundles $\Xi_{e,f,g}:P_{e,g}\to P_{f,g}\otimes_{\mathbb{Z}_2}P_{e,f}$ which makes the following commute:
$$\begin{tikzcd}
\mathcal{PV}_{U,e}^\bullet \arrow[rr, "\Theta_{e,f}"'] \arrow[d, "\Theta_{e,g}"] && \mathcal{PV}_{V,f}^\bullet\otimes_{\mathbb{Z}_2}P_{e,f} \arrow[d, "\Theta_{f,g}\otimes\mathrm{id}_{P_{e,f}}"'] \\
\mathcal{PV}_{W,g}^\bullet\otimes_{\mathbb{Z}_2}P_{e,g} \arrow[r, "\mathrm{id}\otimes\Xi_{e,f,g}"] & & \mathcal{PV}_{W,g}^\bullet\otimes_{\mathbb{Z}_2}P_{f,g}\otimes_{\mathbb{Z}_2}P_{e,f}.
\end{tikzcd}$$

(d) In Question 1.1(d), we call $(V,f,j),(W,g,k)$ *compatible* if locally near each point of $\mathrm{Crit}(f)$ in V there exist holomorphic $\Phi:V\to W$ with $\Phi\circ j=k$ and $f+I_{\mathrm{d}f}^2=g\circ\Phi+I_{\mathrm{d}f}^2$, where $I_{\mathrm{d}f}$ is the ideal of holomorphic functions on V vanishing on $\mathrm{Crit}(f)$. Compatibility is an equivalence relation.

If (V,f,j) and (W,g,k) are compatible there is a natural isomorphism

$$\Delta_{f,g}:j^*(\mathcal{PV}_{V,f}^\bullet)\longrightarrow k^*(\mathcal{PV}_{W,g}^\bullet)\otimes_{\mathbb{Z}_2}Q_{f,g},$$

where $Q_{f,g}\to X$ is a principal $\mathbb{Z}_2$-bundle similar to $P_{f,g}$ in (c).

These $\Delta_{f,g}$ are functorial, in that if $(U,e,i),(V,f,j),(W,g,k)$ are compatible we have a canonical isomorphism $\Gamma_{e,f,g}:Q_{e,g}\to Q_{f,g}\otimes_{\mathbb{Z}_2}Q_{e,f}$ of principal $\mathbb{Z}_2$-bundles over X, and a commutative diagram in $\mathrm{Perv}(X)$:

$$\begin{tikzcd}
i^*(\mathcal{PV}_{U,e}^\bullet) \arrow[rr, "\Delta_{e,f}"'] \arrow[d, "\Delta_{e,g}"] && j^*(\mathcal{PV}_{V,f}^\bullet)\otimes_{\mathbb{Z}_2}Q_{e,f} \arrow[d, "\Delta_{f,g}\otimes\mathrm{id}_{Q_{e,f}}"'] \\
k^*(\mathcal{PV}_{W,g}^\bullet)\otimes_{\mathbb{Z}_2}Q_{e,g} \arrow[rr, "\mathrm{id}\otimes\Gamma_{e,f,g}"] && k^*(\mathcal{PV}_{W,g}^\bullet)\otimes_{\mathbb{Z}_2}Q_{f,g}\otimes_{\mathbb{Z}_2}Q_{e,f}.
\end{tikzcd}$$

Here in part (c), passing from $f:V\to\mathbb{C}$ to $g=f\boxplus z_1^2+\cdots+z_n^2:V\times\mathbb{C}^n\to\mathbb{C}$ is an important idea in singularity theory, as in Arnold et al. [1] for instance. It is known as *stabilization*, and f and g are called *stably equivalent*. So, Question 1.1(c) involves the behaviour of perverse sheaves of vanishing cycles behave under stabilization.

We also generalize our results to Saito's mixed Hodge modules. Mixed Hodge modules are perverse sheaves with extra Hodge-theoretic information. The generalizations of our main theorems from perverse sheaves to mixed Hodge modules are more-or-less immediate, and are included in Theorems 3.1, 4.1, 5.2 and 6.7. One advantage of working with mixed Hodge modules rather than perverse sheaves is that the cohomology groups of the former admit weight polynomials which behave motivically (additively over strata), which is not true for Poincaré polynomials of perverse sheaves. Thus, this allows one to do computations over stratifications, restoring the advantage of the Behrend function approach.

These questions arose in the authors' work on categorification of Donaldson–Thomas invariants of Calabi–Yau 3-folds in algebraic geometry [6,7], which was inspired by earlier work of Behrend and Fantechi [4], Dimca and Szendrői [9], and Kontsevich and Soibelman [15]. Let Z be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathcal{M}$ a proper moduli $\mathbb{C}$-scheme of stable coherent sheaves on Z. As in Thomas [29], one can define a virtual cycle $[\mathcal{M}]^{\mathrm{virt}}\in A_0(\mathcal{M})$ for $\mathcal{M}$, and the *Donaldson–Thomas invariant* is $DT(\mathcal{M})=\int_{[\mathcal{M}]^{\mathrm{virt}}}1\in\mathbb{Z}$. Behrend [2] showed that we may write $DT(\mathcal{M})$ as a weighted Euler characteristic $DT(\mathcal{M})=\chi(\mathcal{M},\nu_{\mathcal{M}})$ where $\nu_{\mathcal{M}}$ is the *Behrend function*, a $\mathbb{Z}$-valued constructible function on the $\mathbb{C}$-scheme $\mathcal{M}$.

Joyce and Song [11, Th. 5.4] (see also [6]) showed that we may cover $\mathcal{M}$ by complex analytic open neighbourhoods $\mathcal{U}\subseteq\mathcal{M}$ with isomorphisms of complex analytic spaces $\mathcal{U}\cong\mathrm{Crit}(f)$, for $f:V\to\mathbb{C}$ a holomorphic function on a complex manifold V. Thus we have a perverse sheaf of vanishing cycles $\mathcal{P}_{\mathcal{U}}^\bullet:=\mathcal{PV}_{V,f}^\bullet$ on

$\mathcal{U} \subseteq \mathcal{M}$. By Behrend [2, §1.2], the pointwise Euler characteristics of $\mathcal{P}^\bullet_{\mathcal{U}}$ satisfy $\chi\bigl(\mathcal{H}^*(\mathcal{P}^\bullet_{\mathcal{U}})_x\bigr) = \nu_{\mathcal{M}}(x)$ for $x \in \mathcal{U}$.

Joyce and Song [11, Question 5.7(a)] asked whether it is possible to glue the perverse sheaves $\mathcal{P}^\bullet_{\mathcal{U}_i}$ for an open cover $\mathcal{U}_i \subseteq \mathcal{M}$ for $i \in I$ to obtain a canonical global perverse sheaf $\mathcal{P}^\bullet_{\mathcal{M}}$ on $\mathcal{M}$, whose hypercohomology $\mathbb{H}^\bullet(\mathcal{P}^\bullet_{\mathcal{M}})$ would then satisfy $\chi\bigl(\mathbb{H}^\bullet(\mathcal{P}^\bullet_{\mathcal{M}})\bigr) = DT(\mathcal{M})$, so that $\mathcal{P}^\bullet_{\mathcal{M}}$ is a 'categorification' of the Behrend function $\nu_{\mathcal{M}}$, and the graded vector space $\mathbb{H}^\bullet(\mathcal{P}^\bullet_{\mathcal{M}})$ is a 'categorification' of the Donaldson–Thomas invariant $DT(\mathcal{M})$. We will show in [6] that the answer to this is yes, provided we choose extra 'orientation data' on $\mathcal{M}$, following Kontsevich and Soibelman [14, §5]. In [7] we also study a related categorification problem for intersections of complex Lagrangians in complex symplectic manifolds. Both of [6, 7] rely on the results of this paper.

To carry out this categorification programme, given open $\mathcal{U}_i, \mathcal{U}_j \subseteq \mathcal{M}$ with isomorphisms $\mathcal{U}_i \cong \mathrm{Crit}(f_i)$, $\mathcal{U}_j \cong \mathrm{Crit}(f_j)$ for holomorphic $f_i : V_i \to \mathbb{C}$ and $f_j : V_j \to \mathbb{C}$, we have to understand whether the perverse sheaves $\mathcal{P}^\bullet_{\mathcal{U}_i} = \mathcal{PV}^\bullet_{V_i,f_i}$ on $\mathcal{U}_i$ and $\mathcal{P}^\bullet_{\mathcal{U}_j} = \mathcal{PV}^\bullet_{V_j,f_j}$ on $\mathcal{U}_j$ are isomorphic over $\mathcal{U}_i \cap \mathcal{U}_j$, and if so, whether the isomorphism is canonical, for only then can we hope to glue the $\mathcal{P}^\bullet_{\mathcal{U}_i}$ for $i \in I$ to make $\mathcal{P}^\bullet_{\mathcal{M}}$. Studying these issues led to this paper.

While writing this paper, the authors learned that Y.-H. Kiem and J. Li have independently obtained some related results.

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2 Background material

Sections 2.1–2.6 recall some definitions and useful results about perverse sheaves. We restrict to perverse sheaves with $\mathbb{Q}$-coefficients. A good introductory reference on perverse sheaves is Dimca [8]. Three other books are Kashiwara and Schapira [12], Schürmann [28], and Hotta, Tanisaki and Takeuchi [10]. Massey [17] and Rietsch [21] are surveys on perverse sheaves, and Beilinson, Bernstein and Deligne [5] is an important primary source. Section 2.7 briefly discusses mixed Hodge modules, following Saito [22–25].

2.1 Constructible sheaves and constructible complexes

We begin by discussing constructible complexes, following Dimca [8, §2–§4].

Definition 2.1. Let X be a complex analytic space. Consider sheaves of $\mathbb{Q}$-vector spaces $\mathcal{A}$ on X. A sheaf $\mathcal{A}$ is called *constructible* if there is a locally finite stratification $X = \coprod_{j \in J} X_j$ of X in the complex analytic topology, such that $\mathcal{A}|_{X_j}$ is a $\mathbb{Q}$-local system for all $j \in J$, and all the stalks $\mathcal{A}_x$ for $x \in X$ are finite-dimensional $\mathbb{Q}$-vector spaces.

Write $D(X)$ for the derived category of complexes $\mathcal{A}^\bullet$ of sheaves of $\mathbb{Q}$-vector spaces on X. Write $D^b_c(X)$ for the full subcategory of bounded complexes $\mathcal{A}^\bullet$ in $D(X)$ whose cohomology sheaves $\mathcal{H}^m(\mathcal{A}^\bullet)$ are constructible for all $m \in \mathbb{Z}$. Then $D(X), D^b_c(X)$ are triangulated categories. An example of a constructible complex on X is the *constant sheaf* $\mathbb{Q}_X$ on X with fibre $\mathbb{Q}$ at each point.

Grothendieck's "six operations on sheaves" $f^*, f^!, Rf_*, Rf_!, \mathcal{RH}om, \overset{L}{\otimes}$ act on $D(X)$, and under extra conditions also act on $D^b_c(X)$. That is, if $f : X \to Y$ is a morphism of complex analytic spaces, then we have two different pullback functors $f^*, f^! : D(Y) \to D(X)$, which also map $D^b_c(Y) \to D^b_c(X)$. Here f^* is called the *inverse image* [8, §2.3], and $f^!$ the *exceptional inverse image* [8, §3.2].

We also have two different pushforward functors $Rf_*, Rf_! : D(X) \to D(Y)$, where Rf_* is called the *direct image* [8, §2.3] and is right adjoint to $f^* : D(Y) \to D(X)$, and $Rf_!$ is called the *direct image with proper supports* [8, §2.3] and is left adjoint to $f^! : D(Y) \to D(X)$.

For constructible complexes in algebraic geometry, if $f : X \to Y$ is a morphism of complex algebraic varieties, then $Rf_*, Rf_!$ also map $D^b_c(X) \to D^b_c(Y)$. However, in the complex analytic context in which we work in this paper, $Rf_*, Rf_!$ in general *do not map* $D^b_c(X) \to D^b_c(Y)$ without extra assumptions on f. For example, if $f : X \to Y$ is a *proper* morphism of complex analytic spaces, then $Rf_*, Rf_!$ map $D^b_c(X) \to D^b_c(Y)$. More generally, if $\mathcal{A}^\bullet \in D^b_c(X)$ and f is proper on the support $\operatorname{supp} \mathcal{A}^\bullet$ then $Rf_*(\mathcal{A}^\bullet), Rf_!(\mathcal{A}^\bullet) \in D^b_c(Y)$.

The only place below where we use $Rf_*(\mathcal{A}^\bullet), Rf_!(\mathcal{A}^\bullet)$ without f being proper on $\operatorname{supp} \mathcal{A}^\bullet$ is the definition of the nearby cycle functor $\psi_f : D^b_c(X) \to D^b_c(X_0)$ in Definition 2.11, and Dimca [8, p. 103] proves ψ_f maps to $D^b_c(X_0) \subset D(X_0)$. So we will ignore this issue.

For $\mathcal{A}^\bullet, \mathcal{B}^\bullet$ in $D^b_c(X)$, we may form their *left derived tensor product* $\mathcal{A}^\bullet \overset{L}{\otimes} \mathcal{B}^\bullet$ in $D^b_c(X)$. Given complex analytic spaces X, Y and objects $\mathcal{A}^\bullet$ in $D^b_c(X)$ and $\mathcal{B}^\bullet$ in $D^b_c(Y)$, we define $\mathcal{A}^\bullet \overset{L}{\boxtimes} \mathcal{B}^\bullet = \pi_X^*(\mathcal{A}^\bullet) \overset{L}{\otimes} \pi_Y^*(\mathcal{B}^\bullet)$ in $D^b_c(X \times Y)$, where $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are the projections.

If X is a complex analytic space, there is a functor $\mathbb{D}_X : D^b_c(X) \to D^b_c(X)^{\mathrm{op}}$ with $\mathbb{D}_X \circ \mathbb{D}_X \cong \mathrm{id} : D^b_c(X) \to D^b_c(X)$, called *Verdier duality*.

Here are some properties of all these:

Theorem 2.2. *In the following, W, X, Y, Z are complex analytic spaces, and e, f, g, h, i are morphisms of complex analytic spaces, and all isomorphisms '$\cong$' of functors or objects are canonical.*

(i) *For $f : X \to Y$ and $g : Y \to Z$, there are natural isomorphisms of functors*

$$R(g \circ f)_* \cong Rg_* \circ Rf_*, \qquad R(g \circ f)_! \cong Rg_! \circ Rf_!,$$
$$(g \circ f)^* \cong f^* \circ g^*, \qquad (g \circ f)^! \cong f^! \circ g^!.$$

(ii) *If $f : X \to Y$ is proper then $Rf_* \cong Rf_!$.*

(iii) *If $i : X \hookrightarrow Y$ is inclusion of an open subset then $i^* \cong i^!$.*

(iv) *If $f : X \to Y$ then $Rf_! \cong \mathbb{D}_Y \circ Rf_* \circ \mathbb{D}_X$ and $f^! \cong \mathbb{D}_X \circ f^* \circ \mathbb{D}_Y$.*

(v) *If V is a complex manifold then $\mathbb{D}_V(\mathbb{Q}_V) \cong \mathbb{Q}_V[2 \dim V]$.*

If $i : X \hookrightarrow Y$ is inclusion of an open subset then we will write '$|_X$' for $i^* : D^b_c(Y) \to D^b_c(X)$, so that $\mathcal{A}^\bullet|_X = i^*(\mathcal{A}^\bullet) \in D^b_c(X)$ for $\mathcal{A}^\bullet \in D^b_c(Y)$. Theorem 2.2(iii) shows we could have used $i^! : D^b_c(Y) \to D^b_c(X)$ instead.

2.2 Perverse sheaves

Next we explain perverse sheaves, following Dimca [8, §5].

Definition 2.3. Let X be a complex analytic space, and for each point $x \in X$, let $i_x : * \to X$ map $i_x : * \mapsto x$. If $\mathcal{A}^\bullet \in D^b_c(X)$, then the *support* $\operatorname{supp}^m \mathcal{A}^\bullet$ and *cosupport* $\operatorname{cosupp}^m \mathcal{A}^\bullet$ of $\mathcal{H}^m(\mathcal{A}^\bullet)$ for $m \in \mathbb{Z}$ are

$$\operatorname{supp}^m \mathcal{A}^\bullet = \overline{\{x \in X : \mathcal{H}^m(i_x^*(\mathcal{A}^\bullet)) \neq 0\}},$$
$$\operatorname{cosupp}^m \mathcal{A}^\bullet = \overline{\{x \in X : \mathcal{H}^m(i_x^!(\mathcal{A}^\bullet)) \neq 0\}},$$

where $\overline{\{\cdots\}}$ means the closure in X. Then $\operatorname{cosupp}^m \mathcal{A}^\bullet = \operatorname{supp}^{-m} \mathbb{D}_X(\mathcal{A}^\bullet)$. We call $\mathcal{A}^\bullet$ *perverse*, or a *perverse sheaf*, if for all $m \in \mathbb{Z}$ we have $\dim \operatorname{supp}^{-m} \mathcal{A}^\bullet \leqslant m$ and $\dim \operatorname{cosupp}^m \mathcal{A}^\bullet \leqslant m$, where by convention $\dim \emptyset = -\infty$. (Here, and throughout, all dimensions are complex dimensions.) Note that perverse sheaves are actually complexes of sheaves, not sheaves. Write $\operatorname{Perv}(X)$ for the full subcategory of perverse sheaves in $D^b_c(X)$. Then $\operatorname{Perv}(X)$ is an abelian category, which is the heart of a t-structure (the *perverse t-structure*) on $D^b_c(X)$.

Perverse sheaves have the following properties:

Theorem 2.4. *In the following, X, Y are complex analytic spaces, and f, i are morphisms of complex analytic spaces.*

(a) *Verdier duality $\mathbb{D}_X : D^b_c(X) \to D^b_c(X)$ maps $\operatorname{Perv}(X) \to \operatorname{Perv}(X)$.*

(b) *If $i : X \hookrightarrow Y$ is inclusion of a closed subspace, and hence proper, then Ri_* and $Ri_!$ (which are naturally isomorphic) map $\operatorname{Perv}(X) \to \operatorname{Perv}(Y)$. Write $\operatorname{Perv}(Y)_X$ for the full subcategory of objects in $\operatorname{Perv}(Y)$ supported on X. Then $Ri_* \cong Ri_!$ are equivalences of categories $\operatorname{Perv}(X) \xrightarrow{\sim} \operatorname{Perv}(Y)_X$. The restricted functors $i^*|_{\operatorname{Perv}(Y)_X}, i^!|_{\operatorname{Perv}(Y)_X}$ map $\operatorname{Perv}(Y)_X \to \operatorname{Perv}(X)$, are naturally isomorphic, and are quasi-inverses for $Ri_*, Ri_! : \operatorname{Perv}(X) \to \operatorname{Perv}(Y)_X$.*

(c) *If $i : X \hookrightarrow Y$ is inclusion of an open subspace then $i^* = |_X$ and $i^!$ (which are naturally isomorphic) map $\operatorname{Perv}(Y) \to \operatorname{Perv}(X)$.*

(d) $\overset{L}{\boxtimes} : D^b_c(X) \times D^b_c(Y) \to D^b_c(X \times Y)$ *maps $\operatorname{Perv}(X) \times \operatorname{Perv}(Y) \to \operatorname{Perv}(X \times Y)$.*

(e) *Let V be a complex manifold. Then $\mathbb{Q}_V[\dim V]$ is perverse, where $\mathbb{Q}_V$ is the constant sheaf on V with fibre $\mathbb{Q}$, and $[\dim V]$ means shift by $\dim V$ in the triangulated category $D^b_c(X)$.*

The next result is proved by Beilinson, Bernstein and Deligne [5, Cor. 2.1.23, §2.2.19, & Th. 3.2.4] in the algebraic case and by Kashiwara and Schapira [12, Th. 10.2.9] in the analytic case; also Hotta et al. [10, Prop. 8.1.26] prove part (i). The analogue for $D^b_c(X)$ or $D(X)$ rather than $\operatorname{Perv}(X)$ is false.

Theorem 2.5. *Let X be a complex analytic space. Then for each open $U \subseteq X$ we have an abelian category* $\mathrm{Perv}(U)$*, and for each inclusion of open sets $V \subseteq U \subseteq X$ there is a functor $|_V : \mathrm{Perv}(U) \to \mathrm{Perv}(V)$ by Theorem* 2.4(c)*. All this data forms a **stack** (a kind of sheaf of categories) on the topological space X.*

This implies that if $\{U_i : i \in I\}$ is an open cover of X, then:

(i) *Suppose $\mathcal{P}^\bullet, \mathcal{Q}^\bullet \in \mathrm{Perv}(X)$, and we are given $\alpha_i : \mathcal{P}^\bullet|_{U_i} \to \mathcal{Q}^\bullet|_{U_i}$ in $\mathrm{Perv}(U_i)$ for all $i \in I$ with $\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$ on $U_{ij} = U_i \cap U_j$ for all $i, j \in I$. Then there exists a unique $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ in $\mathrm{Perv}(X)$ with $\alpha_i = \alpha|_{U_i}$ for all $i \in I$.*

(ii) *Suppose we are given $\mathcal{P}_i^\bullet \in \mathrm{Perv}(U_i)$ for all $i \in I$ and isomorphisms $\alpha_{ij} : \mathcal{P}_i^\bullet|_{U_{ij}} \to \mathcal{P}_j^\bullet|_{U_{ij}}$ for all $i, j \in I$ with $\alpha_{jk}|_{U_{ijk}} \circ \alpha_{ij}|_{U_{ijk}} \circ \alpha_{ik}|_{U_{ijk}}$ on $U_{ijk} = U_i \cap U_j \cap U_k$ for all $i, j, k \in I$, and $\alpha_{ii} = \mathrm{id}_{\mathcal{P}_i^\bullet}$. Then there exists $\mathcal{P}^\bullet$ in $\mathrm{Perv}(X)$, unique up to canonical isomorphism, with isomorphisms $\beta_i : \mathcal{P}^\bullet|_{U_i} \to \mathcal{P}_i^\bullet$ for each $i \in I$, satisfying $\alpha_{ij} \circ \beta_i|_{U_{ij}} = \beta_j|_{U_{ij}}$ for all $i, j \in I$.*

Corollary 2.6. *Let X be a complex analytic space and $\alpha, \beta : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be morphisms in $\mathrm{Perv}(X)$. Then for each $m \in \mathbb{Z}$ and $x \in X$ we have cohomology sheaves $\mathcal{H}^m(\mathcal{P}^\bullet), \mathcal{H}^m(\mathcal{Q}^\bullet)$, which are constructible sheaves on X, and stalks $\mathcal{H}^m(\mathcal{P}^\bullet)_x, \mathcal{H}^m(\mathcal{Q}^\bullet)_x$ at x, which are finite-dimensional $\mathbb{Q}$-vector spaces. So α, β induce morphisms $\mathcal{H}^m(\alpha), \mathcal{H}^m(\beta) : \mathcal{H}^m(\mathcal{P}^\bullet) \to \mathcal{H}^m(\mathcal{Q}^\bullet)$, and linear maps $\mathcal{H}^m(\alpha)_x, \mathcal{H}^m(\beta)_x : \mathcal{H}^m(\mathcal{P}^\bullet)_x \to \mathcal{H}^m(\mathcal{Q}^\bullet)_x$ on stalks.*

Suppose $\mathcal{H}^m(\alpha)_x = \mathcal{H}^m(\beta)_x$ for all $m \in \mathbb{Z}$ and $x \in X$. Then $\alpha = \beta$.

Proof. Suppose $\mathcal{H}^m(\alpha)_x = \mathcal{H}^m(\beta)_x$ for all $m \in \mathbb{Z}$ and $x \in X$. This forces $\alpha_x = \beta_x$, as α_x, β_x are supported at a point. Since α_x, β_x are the germs of α, β at x, $\alpha_x = \beta_x$ implies that for each $x \in X$ we can choose an open $x \in U_x \subseteq X$ with $\alpha|_{U_x} = \beta|_{U_x}$. Then $\{U_x : x \in X\}$ is an open cover of X. The uniqueness of α in Theorem 2.5(i) now proves $\alpha = \beta$. $\square$

2.3 Milnor fibres

The theory of nearby and vanishing cycles for perverse sheaves in §2.4–§2.5 is a generalization of the classical theory of Milnor fibres, so we explain this first.

Definition 2.7. Suppose V is a complex manifold, $f : V \to \mathbb{C}$ a holomorphic function, and $x \in V$. Let $d(,)$ be a metric on V near x. For $\delta, \epsilon > 0$, consider the holomorphic map

$$\Phi_{f,x} : \{y \in V : d(x,y) < \delta,\ 0 < |f(y) - f(x)| < \epsilon\} \longrightarrow \{z \in \mathbb{C} : 0 < |z| < \epsilon\}$$

given by $\Phi_{f,x}(y) = f(y) - f(x)$. Milnor [18] shows that $\Phi_{f,x}$ is a locally trivial smooth fibration provided $0 < \epsilon \ll \delta \ll 1$. The *Milnor fibre* $MF_f(x)$ is the fibre of $\Phi_{f,x}$. It is a noncompact complex manifold of complex dimension $\dim V - 1$, but the complex structure depends on choices of d, ϵ, δ. The underlying real manifold is independent of choices up to diffeomorphism.

Parallel transport around the circle about 0 in $\{z \in \mathbb{C} : 0 < |z| < \epsilon\}$ induces a monodromy transformation $\mu_f : MF_f(x) \to MF_f(x)$, a diffeomorphism defined uniquely up to isotopy.

For applications to perverse sheaves, we are mainly interested in the rational cohomology $H^*\bigl(MF_f(x);\mathbb{Q}\bigr)$ of the Milnor fibre. The monodromy transformation μ_f induces an isomorphism $\mu_{f*} : H^*\bigl(MF_f(x);\mathbb{Q}\bigr) \to H^*\bigl(MF_f(x);\mathbb{Q}\bigr)$, which is independent of choices. It is often convenient to work with the *reduced cohomology* $\tilde H^*\bigl(MF_f(x);\mathbb{Q}\bigr)$, which may be defined by the long exact sequence

$$\cdots \to H^i(*;\mathbb{Q}) \xrightarrow{\pi^*} H^i\bigl(MF_f(x);\mathbb{Q}\bigr) \to \tilde H^i\bigl(MF_f(x);\mathbb{Q}\bigr) \to H^{i+1}(*;\mathbb{Q}) \to \cdots,$$

where $*$ is the point and $\pi : MF_f(x) \to *$ the projection. The monodromy operator μ_{f*} is also defined on $\tilde H^*\bigl(MF_f(x);\mathbb{Q}\bigr)$.

Example 2.8. (a) Suppose V is a complex manifold, $f : V \to \mathbb{C}$ is holomorphic, and $x \in V$ is not a critical value of f, that is, $\mathrm{d}f|_x \neq 0$. Then $MF_f(x)$ is diffeomorphic to an open ball in $\mathbb{C}^{\dim V-1}$, and $\tilde H^*\bigl(MF_f(x);\mathbb{Q}\bigr) = 0$.

(b) Define $f : \mathbb{C}^n \to \mathbb{C}$ by $f(z_1,\ldots,z_n) = z_1^2 + \cdots + z_n^2$ for $n > 1$. Then the Milnor fibre $MF_f(0)$ is diffeomorphic to $T^*\mathcal{S}^{n-1}$, and so homotopic to $\mathcal{S}^{n-1}$, which gives

$$H^i\bigl(MF_f(0);\mathbb{Q}\bigr) \cong \begin{cases} \mathbb{Q}, & i = 0, n-1, \\ 0, & \text{otherwise,} \end{cases} \qquad \tilde H^i\bigl(MF_f(0);\mathbb{Q}\bigr) \cong \begin{cases} \mathbb{Q}, & i = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the isomorphism $\tilde H^{n-1}\bigl(MF_f(0);\mathbb{Q}\bigr) \cong \mathbb{Q}$ is determined by a choice of orientation for $\mathcal{S}^{n-1}$. This is related to the principal $\mathbb{Z}_2$-bundle $P_{f,g}$ in Theorem 5.2, which parametrizes choices of orientation for a quadratic form q locally isomorphic to $z_1^2 + \cdots + z_n^2$.

(c) Suppose $f : \mathbb{C}^n \to \mathbb{C}$ has an isolated critical point at $x = 0$. Then Milnor [18] shows that the Milnor fibre $MF_f(0)$ is homotopic to a bouquet $\mathcal{S}^{n-1}\vee\cdots\vee\mathcal{S}^{n-1}$ of $m(f)$ spheres $\mathcal{S}^{n-1}$, where $m(f)$ is the *Milnor number* of f at 0. Thus $\tilde H^i\bigl(MF_f(0);\mathbb{Q}\bigr)$ is $\mathbb{Q}^{m(f)}$ if $i = n-1$ and 0 otherwise.

Here is the Thom–Sebastiani Theorem, proved by Thom and Sebastiani [27] for isolated singularities, and by Sakamoto [26] in the general case. Related results about $f \boxplus g$ are also sometimes called Thom–Sebastiani theorems.

Theorem 2.9. *Let $f : \mathbb{C}^m \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}$ be holomorphic, and define $f \boxplus g : \mathbb{C}^{m+n} \to \mathbb{C}$ by*

$$(f \boxplus g)(x_1,\ldots,x_m,y_1,\ldots,y_n) = f(x_1,\ldots,x_m) + g(y_1,\ldots,y_n).$$

*Then the Milnor fibre $MF_{f\boxplus g}(0)$ is homotopic to the **join** $MF_f(0) * MF_g(0)$ of $MF_f(0)$ and $MF_g(0)$. It follows that the reduced cohomology groups satisfy*

$$\tilde H^i\bigl(MF_{f\boxplus g}(0);\mathbb{Q}\bigr) \cong \bigoplus_{j,k:i=j+k+1} \tilde H^j\bigl(MF_f(0);\mathbb{Q}\bigr) \otimes_{\mathbb{Q}} \tilde H^k\bigl(MF_g(0);\mathbb{Q}\bigr). \quad (2.1)$$

Furthermore, the monodromy operator $\mu_{(f\boxplus g)*}$ on $\tilde H^*\bigl(MF_{f\boxplus g}(0);\mathbb{Q}\bigr)$ is identified with $\mu_{f*} \otimes \mu_{g*}$ under (2.1).

Local biholomorphisms of V induce diffeomorphisms of Milnor fibres:

Definition 2.10. Let V, W be complex manifolds, $f : V \to \mathbb{C}$, $g : W \to \mathbb{C}$ be holomorphic functions, $x \in X$, and $\Phi : V \to W$ be a local biholomorphism defined near x in V with $\Phi(x) = y$ and $g \circ \Phi = f$. We will define a diffeomorphism of Milnor fibres $\Phi|_{MF_f(x)} : MF_f(x) \to MF_g(y)$.

Choose a metric d on V near x, and let $d' = \Phi_*(d)$ be the pushforward metric on W near y. Then for $0 < \epsilon \ll \delta \ll 1$, we may choose particular models for the Milnor fibres

$$\begin{aligned} MF_f(x)^{d,\epsilon,\delta} &= \{v \in V : d(x,v) < \delta,\ \ f(v) - f(x) = \epsilon\}, \\ MF_g(y)^{d',\epsilon,\delta} &= \{w \in W : d'(y,w) < \delta,\ \ g(w) - g(y) = \epsilon\}. \end{aligned}$$

Then $\Phi|_{MF_f(x)^{d,\epsilon,\delta}} : MF_f(x)^{d,\epsilon,\delta} \to MF_g(y)^{d',\epsilon,\delta}$ is a diffeomorphism.

Of course, there are many different models for the Milnor fibre $MF_f(x)$ as a subset of V, by choosing different metrics d and $0 < \epsilon \ll \delta \ll 1$. But these different models $MF_f(x), MF_f(x)'$ are all diffeomorphic, with the diffeomorphisms $MF_f(x) \to MF_f(x)'$ unique up to isotopy. So, given arbitrary models $MF_f(x), MF_g(y)$ for the Milnor fibres of f, g at x, y, we may define a diffeomorphism $\Phi|_{MF_f(x)} : MF_f(x) \to MF_g(y)$ as the composition

$$MF_f(x) \xrightarrow{\alpha} MF_f(x)^{d,\epsilon,\delta} \xrightarrow{\Phi|_{MF_f(x)^{d,\epsilon,\delta}}} MF_g(y)^{d',\epsilon,\delta} \xrightarrow{\beta} MF_g(y),$$

where α, β are diffeomorphisms between the different models of Milnor fibres, and are natural up to isotopy.

Thus $\Phi|_{MF_f(x)} : MF_f(x) \to MF_g(y)$ is a diffeomorphism, which is unique up to isotopy. In particular, the induced morphism of reduced cohomology $(\Phi|_{MF_f(x)})_* : \tilde{H}^*(MF_f(x), \mathbb{Q}) \to \tilde{H}^*(MF_g(y), \mathbb{Q})$ is independent of choices.

2.4 Nearby cycles and vanishing cycles

We explain nearby cycles and vanishing cycles, as in Dimca [8, §4.2].

Definition 2.11. Let X be a complex analytic space, and $f : X \to \mathbb{C}$ a holomorphic function. Define $X_0 = f^{-1}(0)$, as a complex analytic space, and $X^* = X \setminus X_0$. Consider the commutative diagram:

$$\begin{tikzcd} X_0 \arrow[r, "i"'] \arrow[d, "f"] & X \arrow[d, "f"] & X^* \arrow[l, "j"] \arrow[d, "f"] & \widetilde{X^*} \arrow[l, "p"] \arrow[d, "\tilde{f}"] \arrow[ll, bend right, "\pi"'] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^* \arrow[l] & \widetilde{\mathbb{C}^*}. \arrow[l, "\rho"'] \end{tikzcd}$$

Here $i : X_0 \hookrightarrow X$, $j : X^* \hookrightarrow X$ are the inclusions, $\rho : \widetilde{\mathbb{C}^*} \to \mathbb{C}^*$ is the universal cover of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\widetilde{X^*} = X^* \times_{f,\mathbb{C}^*,\rho} \widetilde{\mathbb{C}^*}$ the corresponding cover of

X^*, with covering map $p : \widetilde{X^*} \to X^*$, and $\pi = j \circ p$. The *nearby cycle functor* $\psi_f : D^b_c(X) \to D^b_c(X_0)$ is $\psi_f = i^* \circ R\pi_* \circ \pi^*$.

There is a natural transformation $\Xi : i^* \Rightarrow \psi_f$ between the functors $i^*, \psi_f : D^b_c(X) \to D^b_c(X_0)$. The *vanishing cycle functor* $\phi_f : D^b_c(X) \to D^b_c(X_0)$ is a functor such that for every $\mathcal{A}^\bullet$ in $D^b_c(X)$ we have a distinguished triangle

$$i^*(\mathcal{A}^\bullet) \xrightarrow{\Xi(\mathcal{A}^\bullet)} \psi_f(\mathcal{A}^\bullet) \longrightarrow \phi_f(\mathcal{A}^\bullet) \xrightarrow{[+1]} i^*(\mathcal{A}^\bullet)$$

in $D^b_c(X_0)$. Following Dimca [8, p. 108], we write ψ^p_f, ϕ^p_f for the shifted functors $\psi_f[-1], \phi_f[-1] : D^b_c(X) \to D^b_c(X_0)$.

The generator of $\mathbb{Z} = \pi_1(\mathbb{C}^*)$ on $\widetilde{\mathbb{C}^*}$ induces a deck transformation $\delta_{\mathbb{C}^*} : \widetilde{\mathbb{C}^*} \to \widetilde{\mathbb{C}^*}$ which lifts to a deck transformation $\delta_{X^*} : \widetilde{X^*} \to \widetilde{X^*}$ with $p \circ \delta_{X^*} = p$ and $\tilde f \circ \delta_{X^*} = \delta_{\mathbb{C}^*} \circ \tilde f$. As in Dimca [8, p. 103, p. 105], we can use δ_{X^*} to define natural transformations $M_{X,f} : \psi^p_f \Rightarrow \psi^p_f$ and $M_{X,f} : \phi^p_f \Rightarrow \phi^p_f$, called *monodromy*. They are generalizations of the monodromy transformations μ_{f*} on $H^*\bigl(MF_f(x);\mathbb{Q}\bigr)$ and $\tilde H^*\bigl(MF_f(x);\mathbb{Q}\bigr)$ in §2.3.

Here are some properties of nearby and vanishing cycles:

Theorem 2.12. (i) *If X is a complex analytic space and $f : X \to \mathbb{C}$ is holomorphic, then $\psi^p_f, \phi^p_f : D^b_c(X) \to D^b_c(X_0)$ both map* $\mathrm{Perv}(X) \to \mathrm{Perv}(X_0)$.

(ii) *If X is a complex analytic space and $f : X \to \mathbb{C}$ is holomorphic, then there are natural isomorphisms $\psi_f \circ \mathbb{D}_X \cong \mathbb{D}_{X_0} \circ \psi_f$ and $\phi_f \circ \mathbb{D}_X \cong \mathbb{D}_{X_0} \circ \phi_f$.*

(iii) *Suppose $\Phi : X \to Y$ is a proper morphism of complex analytic spaces, and $g : Y \to \mathbb{C}$ a holomorphic map. Write $f = g \circ \Phi : X \to \mathbb{C}$, and $X_0 = f^{-1}(0) \subseteq X$, $Y_0 = g^{-1}(0) \subseteq Y$, and $\Phi_0 = \Phi|_{X_0} : X_0 \to Y_0$. Then we have natural isomorphisms of functors $D^b_c(X) \to D^b_c(Y_0)$:*

$$R(\Phi_0)_* \circ \psi_g \cong \psi_f \circ R\Phi_*, \quad R(\Phi_0)_* \circ \phi_g \cong \phi_f \circ R\Phi_*.$$

Note too that $R\Phi_* \cong R\Phi_!$ and $R(\Phi_0)_* \cong R(\Phi_0)_!$, as Φ, Φ_0 are proper.

We can describe how ψ^p_f, ϕ^p_f act on stalks of complexes in terms of Milnor fibres in §2.3, following Dimca [8, Prop. 4.2.2, Ex. 4.2.3 & Ex. 4.2.6].

Theorem 2.13. *Let V be a complex manifold, $f : V \to \mathbb{C}$ be holomorphic, and $\mathcal{A}^\bullet \in D^b_c(V)$. Then for all $x \in V$ with $f(x) = 0$ and all $m \in \mathbb{Z}$ we have a natural isomorphism*

$$\mathcal{H}^m\bigl(\psi^p_f(\mathcal{A}^\bullet)\bigr)_x \cong \mathbb{H}^{m-1}\bigl(MF_f(x), \mathcal{A}^\bullet\bigr). \tag{2.2}$$

Here $MF_f(x) = B_\delta(x) \cap f^{-1}(\epsilon)$ for $0 < \epsilon \ll \delta \ll 1$ is the Milnor fibre of f, as in §2.3, and $\mathbb{H}^*\bigl(MF_f(x), \mathcal{A}^\bullet\bigr)$ is the hypercohomology of $\mathcal{A}^\bullet$ restricted to $MF_f(x)$, and is independent of choices provided ϵ, δ are sufficiently small.

When $\mathcal{A}^\bullet$ is the constant sheaf $\mathbb{Q}_V$, equation (2.2) *becomes*

$$\mathcal{H}^m\bigl(\psi^p_f(\mathbb{Q}_V)\bigr)_x \cong H^{m-1}\bigl(MF_f(x), \mathbb{Q}\bigr), \tag{2.3}$$

and for vanishing cycles ϕ_f^p we have similar isomorphisms

$$\mathcal{H}^m\big(\phi_f^p(\mathbb{Q}_V)\big)_x \cong \tilde{H}^{m-1}\big(MF_f(x),\mathbb{Q}\big). \tag{2.4}$$

Furthermore, the actions of the monodromy operators $M_{V,f}$ on the left hand sides of (2.2)–(2.4) *are identified with the automorphisms of the right hand sides induced by the monodromy transformation $\mu_f : MF_f(x) \to MF_f(x)$.*

2.5 The perverse sheaf of vanishing cycles $\mathcal{PV}_{V,f}^\bullet$

We can now define the main subject of this paper, the perverse sheaf of vanishing cycles $\mathcal{PV}_{V,f}^\bullet$ for a holomorphic function $f : V \to \mathbb{C}$.

Definition 2.14. Let V be a complex manifold, and $f : V \to \mathbb{C}$ a holomorphic function. Write $X = \mathrm{Crit}(f)$, as a closed complex analytic subspace of V. As a map of topological spaces (though not necessarily as a morphism of complex analytic spaces), $f|_X : X \to \mathbb{C}$ is locally constant, whose image $f(X)$ is finite or countable. Thus we have a decomposition $X = \coprod_{c\in f(X)} X_c$, where X_c is the open and closed complex analytic subspace of points x in X with $f(x) = c$.

For each $c \in \mathbb{C}$, write $V_c = f^{-1}(c) \subseteq V$. Then as in §2.4, we have a vanishing cycle functor $\phi_{f-c}^p : \mathrm{Perv}(V) \to \mathrm{Perv}(V_c)$. So we may form $\phi_{f-c}^p(\mathbb{Q}_V[\dim V])$ in $\mathrm{Perv}(V_c)$, since $\mathbb{Q}_V[\dim V] \in \mathrm{Perv}(V)$ by Theorem 2.4(e).

Applying a shift to (2.4) shows that for all $v \in V_c$ and $m \in \mathbb{Z}$ we have

$$\mathcal{H}^m\big(\phi_{f-c}^p(\mathbb{Q}_V[\dim V])\big)_v \cong \tilde{H}^{m-1+\dim V}\big(MF_f(v),\mathbb{Q}\big). \tag{2.5}$$

Suppose $v \in V_c$ is not a critical point. Then Example 2.8(a) shows that $\tilde{H}^*\big(MF_f(v);\mathbb{Q}\big) = 0$, so (2.5) gives $\mathcal{H}^*\big(\phi_{f-c}^p(\mathbb{Q}_V[\dim V])\big)_v = 0$. Therefore $\phi_{f-c}^p(\mathbb{Q}_V[\dim V])$ is supported on the closed subset $X_c = \mathrm{Crit}(f) \cap V_c$ in V_c, where $X_c = \emptyset$ unless $c \in f(X)$. That is, $\phi_{f-c}^p(\mathbb{Q}_V[\dim V])$ lies in $\mathrm{Perv}(V_c)_{X_c}$.

But Theorem 2.4(b) shows that $\mathrm{Perv}(V_c)_{X_c}$ and $\mathrm{Perv}(X_c)$ are equivalent categories, so we may regard $\phi_{f-c}^p(\mathbb{Q}_V[\dim V])$ as a perverse sheaf on X_c. That is, we can consider $\phi_{f-c}^p(\mathbb{Q}_V[\dim V])|_{X_c} = i_{X_c,V_c}^*\big(\phi_{f-c}^p(\mathbb{Q}_V[\dim V])\big)$ in $\mathrm{Perv}(X_c)$, where $i_{X_c,V_c} : X_c \to V_c$ is the inclusion morphism.

As $X = \coprod_{c\in f(X)} X_c$ with each X_c open and closed in X, we have $\mathrm{Perv}(X) = \bigoplus_{c\in f(X)} \mathrm{Perv}(X_c)$. Define the *perverse sheaf of vanishing cycles $\mathcal{PV}_{V,f}^\bullet$ of V, f* in $\mathrm{Perv}(X)$ to be $\mathcal{PV}_{V,f}^\bullet = \bigoplus_{c\in f(X)} \phi_{f-c}^p(\mathbb{Q}_V[\dim V])|_{X_c}$. That is, $\mathcal{PV}_{V,f}^\bullet$ is the unique perverse sheaf on $X = \mathrm{Crit}(f)$ with $\mathcal{PV}_{V,f}^\bullet|_{X_c} = \phi_{f-c}^p(\mathbb{Q}_V[\dim V])|_{X_c}$ for all $c \in f(X)$. Equation (2.5) now gives

$$\mathcal{H}^m\big(\mathcal{PV}_{V,f}^\bullet\big)_x \cong \tilde{H}^{m-1+\dim V}\big(MF_f(x),\mathbb{Q}\big) \quad \text{for all } x \in X,\ m \in \mathbb{Z}. \tag{2.6}$$

For $c \in f(X)$, we have a monodromy operator $M_{V,f-c} : \phi_{f-c}^p(\mathbb{Q}_V[\dim V]) \to \phi_{f-c}^p(\mathbb{Q}_V[\dim V])$, which restricts to $\phi_{f-c}^p(\mathbb{Q}_V[\dim V])|_{X_c}$. Define the *twisted monodromy operator* $\tau_{V,f} : \mathcal{PV}_{V,f}^\bullet \to \mathcal{PV}_{V,f}^\bullet$ by

$$\begin{aligned}\tau_{V,f}|_{X_c} = (-1)^{\dim V} M_{V,f-c}|_{X_c} :&\\ \phi_{f-c}^p(\mathbb{Q}_V[\dim V])|_{X_c} &\longrightarrow \phi_{f-c}^p(\mathbb{Q}_V[\dim V])|_{X_c},\end{aligned} \tag{2.7}$$

for each $c \in f(X)$. Here 'twisted' refers to the sign $(-1)^{\dim V}$ in (2.7). We include this sign change as it makes monodromy act naturally under transformations which change dimension — without it, equations (5.3) and (6.13) below would only commute up to a sign $(-1)^{\dim W - \dim V}$, not commute — and it normalizes the monodromy of any nondegenerate quadratic form to be the identity, as in (2.9). The sign $(-1)^{\dim V}$ also corresponds to the twist '$(\frac{1}{2}\dim V)$' in the definition (2.15) of the mixed Hodge module of vanishing cycles $\mathcal{HV}^\bullet_{V,f}$ in §2.7.

Under Verdier duality, we have $\mathbb{D}_V(\mathbb{Q}_V[\dim V]) \cong \mathbb{Q}_V[\dim V]$ by Theorem 2.2(v), so $\mathbb{D}_{V_c}\big(\phi^p_{f-c}(\mathbb{Q}_V[\dim V])\big) \cong \phi^p_{f-c}(\mathbb{Q}_V[\dim V])$ by Theorem 2.12(ii). Applying $i^*_{X_c,V_c}$ and using $\mathbb{D}_{X_c} \circ i^*_{X_c,V_c} \cong i^!_{X_c,V_c} \circ \mathbb{D}_{V_c}$ by Theorem 2.2(iv) and $i^!_{X_c,V_c} \cong i^*_{X_c,V_c}$ on $\mathrm{Perv}(V_c)_{X_c}$ by Theorem 2.4(b) also gives

$$\mathbb{D}_{X_c}\big(\phi^p_{f-c}(\mathbb{Q}_V[\dim V])|_{X_c}\big) \cong \phi^p_{f-c}(\mathbb{Q}_V[\dim V])|_{X_c}.$$

Summing over all $c \in f(X)$ yields an isomorphism $\mathbb{D}_X(\mathcal{PV}^\bullet_{V,f}) \cong \mathcal{PV}^\bullet_{V,f}$.

There is a 'Thom–Sebastiani Theorem for perverse sheaves', due to Massey [16] and Schürmann [28, Cor. 1.3.4]. Applied to $\mathcal{PV}^\bullet_{V,f}$, it yields:

Theorem 2.15. *Let V, W be complex manifolds and $f : V \to \mathbb{C}$, $g : W \to \mathbb{C}$ be holomorphic, so that $f \boxplus g : V \times W \to \mathbb{C}$ is holomorphic with $(f \boxplus g)(v, w) := f(v) + g(w)$. Set $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as complex analytic subspaces of V, W, so that $\mathrm{Crit}(f \boxplus g) = X \times Y$. Then there is a natural isomorphism*

$$\mathcal{PV}^\bullet_{V\times W, f\boxplus g} \cong \mathcal{PV}^\bullet_{V,f} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{W,g} \qquad \text{in } \mathrm{Perv}(X \times Y),$$

identifying twisted monodromy operators $\tau_{V\times W, f\boxplus g}$ and $\tau_{V,f} \overset{L}{\boxtimes} \tau_{W,g}$ in (2.7).

Note that specializing to cohomology of stalks using (2.6), we recover the classical Thom–Sebastiani Theorem, Theorem 2.9.

Example 2.16. Define $f : \mathbb{C}^n \to \mathbb{C}$ by $f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$ for $n > 1$. Then $\mathrm{Crit}(f) = \{0\}$, so $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} = \phi^p_f(\mathbb{Q}_{\mathbb{C}^n}[n])|_{\{0\}}$ is a perverse sheaf on the point $\{0\}$. Also Example 2.8(b) shows that $MF_f(0) \cong T^*\mathcal{S}^{n-1}$ and gives an expression for $\tilde{H}^*(MF_f(0); \mathbb{Q})$. Hence (2.6) gives

$$\mathcal{H}^m\big(\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}\big)_0 \cong \begin{cases} H^{n-1}(\mathcal{S}^{n-1}, \mathbb{Q}) \cong \mathbb{Q}, & m = 0, \\ 0, & m \neq 0. \end{cases}$$

Therefore we have an isomorphism

$$\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong \mathbb{Q}_{\{0\}}. \tag{2.8}$$

This isomorphism (2.8) is natural up to sign, as it depends on the isomorphism $H^{n-1}(\mathcal{S}^{n-1}, \mathbb{Q}) \cong \mathbb{Q}$, which corresponds to a choice of orientation for $\mathcal{S}^{n-1}$.

One can show that the monodromy transformation $\mu_f : MF_f(0) \to MF_f(0)$ acts on $MF_f(0) \cong T^*\mathcal{S}^{n-1}$ as the map $\mathrm{d}(-1) : T^*\mathcal{S}^{n-1} \to T^*\mathcal{S}^{n-1}$ induced by $-1 : \mathcal{S}^{n-1} \to \mathcal{S}^{n-1}$ mapping $-1 : (x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_n)$. This

multiplies orientations on $\mathcal{S}^{n-1}$ by $(-1)^n$. Thus, $\mu_{f*}: H^{n-1}(\mathcal{S}^{n-1},\mathbb{Q}) \to H^{n-1}(\mathcal{S}^{n-1},\mathbb{Q})$ multiplies by $(-1)^n$. Theorem 2.13 now implies that the monodromy $M_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}$ acts on $\mathcal{H}^*\bigl(\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}\bigr)_0$ as $(-1)^n$. As $\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}$ is supported at 0, this determines the action. Combining this with the sign change $(-1)^{\dim V}$ in (2.7) for $V=\mathbb{C}^n$ shows that the twisted monodromy is

$$\tau_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} = \mathrm{id} : \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \longrightarrow \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}. \tag{2.9}$$

Equations (2.8)–(2.9) also hold for $n=0,1$.

2.6 Symmetries of perverse sheaves of vanishing cycles

To prove our main results, if V,f,X and $\mathcal{PV}^\bullet_{V,f}$ are as in Definition 2.14 and Φ is a symmetry of (V,f) acting trivially on X, we will study the action of Φ on $\mathcal{PV}^\bullet_{V,f}$. We will need the following notation.

Definition 2.17. Let V be a complex manifold, and $f:V\to\mathbb{C}$ a holomorphic function, and set $X=\mathrm{Crit}(f)$, so that Definition 2.14 defines the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{V,f}$ in $\mathrm{Perv}(X)$.

Suppose $g:V\to\mathbb{C}$ is another holomorphic function and $\Phi:V\to V$ is a local biholomorphism defined near X in V with $g\circ\Phi=f$ and $\Phi|_X=\mathrm{id}_X$. This implies that $\mathrm{Crit}(g)=X$ on the domain of Φ, so suppose $\mathrm{Crit}(g)=X$. Define an isomorphism of perverse sheaves $\Phi_*:\mathcal{PV}^\bullet_{V,f}\to\mathcal{PV}^\bullet_{V,g}$ by the commutative diagram for each $c\in f(X)=g(X)$:

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{V,f}|_{X_c} = \phi^p_{f-c}(\mathbb{Q}_V[\dim V])|_{X_c} \arrow[r, "\alpha"'] \arrow[dd, "\Phi_*|_{X_c}"] & (\mathrm{id}_{X_c})_*\circ\phi^p_{f-c}(\mathbb{Q}_V[\dim V])|_{X_c} \arrow[d, equal] \\
& R(\Phi_c)_*\circ\phi^p_{(g-c)\circ\Phi}(\mathbb{Q}_V[\dim V])|_{X_c} \arrow[d, "\beta"'] \\
\mathcal{PV}^\bullet_{V,g}|_{X_c} = \phi^p_{g-c}(\mathbb{Q}_V[\dim V])|_{X_c} & \phi^p_{g-c}\circ R\Phi_*(\mathbb{Q}_V[\dim V])|_{X_c}. \arrow[l, "\gamma"']
\end{tikzcd} \tag{2.10}$$

Here we write Φ_c for the induced map $\Phi|_{f^{-1}(c)}: f^{-1}(c)\to g^{-1}(c)$. The isomorphism α comes from $\mathcal{A}^\bullet\cong(\mathrm{id}_{X_c})_*(\mathcal{A}^\bullet)$ for $\mathcal{A}^\bullet\in\mathrm{Perv}(X_c)$. The top right equality in (2.10) holds as $\Phi_c|_{X_c}=\mathrm{id}_{X_c}$ and $(g-c)\circ\Phi=f-c$. The isomorphism β comes from Theorem 2.12(iii), noting that Φ is proper near X. The isomorphism γ comes from $R\Phi_*(\mathbb{Q}_V[\dim V])\cong\mathbb{Q}_V[\dim V]$ near X, as Φ is a local biholomorphism.

Here is an expression for how Φ_* acts on the stalk of $\mathcal{PV}^\bullet_{V,f}$ at $x\in X$. For each $m\in\mathbb{Z}$ we have a commutative diagram of isomorphisms

$$\begin{tikzcd}
\mathcal{H}^m\bigl(\mathcal{PV}^\bullet_{V,f}\bigr)_x \arrow[r, "(2.6)"'] \arrow[d, "\mathcal{H}^m(\Phi_*)_x"] & \tilde H^{m-1+\dim V}\bigl(MF_f(x),\mathbb{Q}\bigr) \arrow[d, "(\Phi|_{MF_f(x)})_*"'] \\
\mathcal{H}^m\bigl(\mathcal{PV}^\bullet_{V,g}\bigr)_x \arrow[r, "(2.6)"] & \tilde H^{m-1+\dim V}\bigl(MF_g(x),\mathbb{Q}\bigr),
\end{tikzcd} \tag{2.11}$$

where the rows come from (2.6), and $\Phi|_{MF_f(x)}$ is as in Definition 2.10.

These isomorphisms Φ_* are functorial, and commute with monodromy. That is, if $h : V \to \mathbb{C}$ is another holomorphic function and $\Psi : V \to V$ a local biholomorphism defined near X with $h \circ \Psi = g$ and $\Psi|_X = \mathrm{id}_X$, then

$$(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_* : \mathcal{PV}^\bullet_{V,f} \longrightarrow \mathcal{PV}^\bullet_{V,h}.$$

Also $(\Phi^{-1})_* = (\Phi_*)^{-1}$, and $(\mathrm{id}_V)_* = \mathrm{id}$.

Example 2.18. Define $f : \mathbb{C}^n \to \mathbb{C}$ by $f(z_1, \dots, z_n) = z_1^2 + \cdots + z_n^2$, so that $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong \mathbb{Q}_{\{0\}}$ as in Example 2.16. Let $A \in \mathrm{O}(n, \mathbb{C})$ be an orthogonal matrix, so that $A : \mathbb{C}^n \to \mathbb{C}^n$ is a biholomorphism with $f \circ A = f$ and $A|_{\{0\}} = \mathrm{id}_{\{0\}}$. So Definition 2.17 defines an isomorphism

$$A_* : \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \longrightarrow \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}. \tag{2.12}$$

As above, A maps $MF_f(0) \cong T^*\mathcal{S}^{n-1}$ to itself. One can show that the induced action $(A|_{MF_f(0)})_*$ on $\tilde{H}^*\big(MF_f(x), \mathbb{Q}\big)$ is multiplication by $\det A = \pm 1$. Thus (2.11) shows $\mathcal{H}^m(A_*)_0$ acts on $\mathcal{H}^m\big(\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}\big)_0$ by multiplication by $\det A$, for each $m \in \mathbb{Z}$. Since $\mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2}$ is supported at a point, A_* in (2.12) is multiplication by $\det A = \pm 1$.

2.7 Mixed Hodge modules

For a complex analytic space X, let $\mathrm{MHM}(X)$ denote Saito's category [22, 23] of mixed Hodge modules, and $D^b \mathrm{MHM}(X)$ its derived category. Recall that there is a faithful and exact forgetful functor $\mathbf{rat} : \mathrm{MHM}(X) \to \mathrm{Perv}(X)$ to the category of perverse $\mathbb{Q}$-sheaves on X in §2.2, extending to a functor $\mathbf{Rat} : D^b \mathrm{MHM}(X) \to D^b_c(X)$. We spell out the faithfulness of $\mathbf{rat}$, since it will be essential for us: it means that for all $M_1^\bullet, M_2^\bullet \in \mathrm{MHM}(X)$, we have an injection

$$\mathrm{Hom}_{\mathrm{MHM}(X)}\big(M_1^\bullet, M_2^\bullet\big) \hookrightarrow \mathrm{Hom}_{\mathrm{Perv}(X)}\big(\mathbf{rat}\, M_1^\bullet, \mathbf{rat}\, M_2^\bullet\big). \tag{2.13}$$

Thus, a morphism in $\mathrm{MHM}(X)$ is uniquely determined by the underlying morphism of perverse sheaves.

Mixed Hodge modules carry functorial weight filtrations, with graded pieces being objects in the category of pure Hodge modules $\mathrm{HM}(X)$ of [22].

Theorem 2.19. *The categories of mixed Hodge modules for complex analytic spaces have the following properties:*

(i) *There is a duality functor* $\mathbb{D}^H_X : D^b \mathrm{MHM}(X) \to D^b \mathrm{MHM}(X)$ *which preserves the abelian category* $\mathrm{MHM}(X)$*.*

(ii) *There are tensor product and external tensor product functors*

$$\begin{aligned} \overset{L}{\otimes} &: D^b \mathrm{MHM}(X) \times D^b \mathrm{MHM}(X) \longrightarrow D^b \mathrm{MHM}(X), \\ \overset{L}{\boxtimes} &: D^b \mathrm{MHM}(X) \times D^b \mathrm{MHM}(Y) \longrightarrow D^b \mathrm{MHM}(X \times Y), \end{aligned}$$

both mapping $\mathrm{MHM}(-) \times \mathrm{MHM}(-) \to \mathrm{MHM}(-)$*.*

(iii) *For a morphism $f : X \to Y$ of complex analytic spaces, there are pullback functors $f^*, f^! : D^b \operatorname{MHM}(Y) \to D^b \operatorname{MHM}(X)$.*

(iv) *For a closed embedding $i : X \to Y$, there is a pushforward functor $Ri_* \cong Ri_! : D^b \operatorname{MHM}(X) \to D^b \operatorname{MHM}(Y)$ which maps $\operatorname{MHM}(X)$ to $\operatorname{MHM}(Y)$. Its image can be identified with the subcategory of $\operatorname{MHM}(Y)$ of objects supported on X. More generally, for a proper morphism $f : X \to Y$, there are pushforward functors $Rf_*, Rf_! : D^b \operatorname{MHM}(X) \to D^b \operatorname{MHM}(Y)$.*

(v) *All the functors above are compatible with the corresponding functors on constructible complexes, via the forgetful functor* **rat***. They also satisfy all the "six operations" identities.*

(vi) *For smooth X, we have a canonical object $\mathbb{Q}^H_X[\dim X]$ in $\operatorname{MHM}(X)$.*

(vii) *The category of mixed Hodge modules for X a point is canonically equivalent to Deligne's category of mixed Hodge structures.*

(viii) *Mixed Hodge modules form a stack: both objects and morphisms can be glued from compatible data on an open cover.*

Proof. Properties (i)–(vii) are standard and can be found in [23]. Property (viii) is proved in a general context in [24, Cor. 2.3] (in the algebraic category, but the axiomatic properties needed also hold in the analytic case). □

Remark 2.20. The theorem summarizes our notational conventions: Hodge-theoretic objects will be denoted by superscript $(\cdots)^H$ whenever convenient to do so, but not at the expense of readability.

It is important to note that in the analytic context, the pushforward of a mixed Hodge module under an arbitrary open embedding may not exist.

Suppose that $M^\bullet$ is a complex of mixed Hodge modules on a complex space X, and assume that the underlying $\mathbb{Q}$-complex $\mathbf{rat}\, M^\bullet$ has finite-dimensional (hyper)cohomology groups $\mathbb{H}^*(\mathbf{rat}\, M^\bullet)$. Then the hypercohomology $\mathbb{H}^*(M^\bullet)$ carries a functorial mixed Hodge structure, in particular a weight filtration, and therefore has a weight polynomial.

Example 2.21. Let $\pi : \mathbb{P}^1 \to \mathrm{pt}$ be the projection of the projective line to a point. Then in the category MHM(pt), in other words the category of mixed Hodge structures, we have a decomposition $R\pi_*\big(\mathbb{Q}^H_{\mathbb{P}^1}\big) \cong \mathbb{Q}^H_{\mathrm{pt}} \oplus \mathcal{T}[-2]$ of the cohomology of $\mathbb{P}^1$. Here $\mathcal{T}$ is the *Lefschetz Hodge structure*, a one-dimensional pure Hodge structure, which as usual we will denote by $\mathbb{Q}(1)$, and which admits an inverse under the tensor product. Then for any X with structure morphism $\pi : X \to \mathrm{pt}$, and any $M^\bullet \in D^b \operatorname{MHM}(X)$, we denote $M^\bullet(n) = M^\bullet \overset{L}{\otimes} \pi^*\big(\mathbb{Q}(1)^{\otimes^n}\big)$.

Next, we discuss nearby and vanishing cycle functors. To do this in a way consistent with monodromy, we need a mild extension of the category. For a complex analytic space X, denote by $\operatorname{MHM}(X; T_s, N)$ the category of mixed Hodge modules $M^\bullet$ on X with commuting actions of a finite order operator $T_s : M^\bullet \to M^\bullet$ and a locally nilpotent operator $N : M^\bullet \to M^\bullet(-1)$, and

$D^b \operatorname{MHM}(X; T_s, N)$ its derived category, following Saito [25, §4.2]. Theorem 2.19 remains true in this context.

Note that the definition of the external tensor product $\overset{L}{\boxtimes}$ on $\operatorname{MHM}(-; T_s, N)$, including the weight and Hodge filtrations on $M_1^\bullet \overset{L}{\boxtimes} M_2^\bullet$, involves the action of the finite order endomorphism T_s, as in Saito [25, §5.1]. The forgetful functors $\operatorname{MHM}(-; T_s, N) \to \operatorname{MHM}(-)$ do not map $\overset{L}{\boxtimes}$ on $\operatorname{MHM}(-; T_s, N)$ to $\overset{L}{\boxtimes}$ on $\operatorname{MHM}(-)$. We need this special definition of $\overset{L}{\boxtimes}$ on $\operatorname{MHM}(-; T_s, N)$ to correctly state the Thom–Sebastiani theorem for mixed Hodge modules [25, Th. 5.4], and its consequence Theorem 2.23 below.

As in Saito [23], for holomorphic $f : X \to \mathbb{C}$, the perverse nearby and vanishing cycle functors ψ_f^p, ϕ_f^p defined on perverse sheaves in §2.4 lift to functors $\psi_f^{pH}, \phi_f^{pH} : \operatorname{MHM}(X) \to \operatorname{MHM}(X_0; T_s, N)$, where $X_0 = f^{-1}(0)$. The actions of the finite order and nilpotent operators T_s, N are given by the semisimple part of the monodromy operator, and the logarithm of its unipotent part.

Example 2.22. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$. Then $\operatorname{Crit}(f) = \{0\}$, and we obtain an object $\phi_f^{pH}\big(\mathbb{Q}_{\mathbb{C}}^H[1]\big)$ in $\operatorname{MHM}(\mathrm{pt}; T_s, N)$, a one-dimensional mixed Hodge structure with monodromy acting by -1. For $g : \mathbb{C}^2 \to \mathbb{C}$ given by $g(z_1, z_2) = z_1^2 + z_2^2$, it is well known that

$$\phi_{z_1^2+z_2^2}^{pH}\big(\mathbb{Q}_{\mathbb{C}^2}^H[2]\big) \cong \mathbb{Q}(-1),$$

with trivial monodromy action. Applying the Thom–Sebastiani formula for mixed Hodge modules [25, Th. 5.4], we see that

$$\phi_{z^2}^{pH}\big(\mathbb{Q}_{\mathbb{C}}^H[1]\big) \otimes \phi_{z^2}^{pH}\big(\mathbb{Q}_{\mathbb{C}}^H[1]\big) \cong \mathbb{Q}(-1)$$

in the category $\operatorname{MHM}(\mathrm{pt}; T_s, N)$. The objects $\mathbb{Q}(1)$ and $\mathbb{Q}(-1)$ thus admit square roots in this category, which we will denote by $\mathbb{Q}(\frac{1}{2})$ and $\mathbb{Q}(-\frac{1}{2})$, where

$$\phi_{z^2}^{pH}\big(\mathbb{Q}_{\mathbb{C}}^H[1]\big) = \mathbb{Q}(-\tfrac{1}{2}). \tag{2.14}$$

Define an object $\mathbb{Q}(\frac{n}{2}) \in \operatorname{MHM}(\mathrm{pt}; T_s, N)$ for each $n \in \mathbb{Z}$ by $\mathbb{Q}(\frac{n}{2}) = \mathbb{Q}(\frac{1}{2})^{\otimes^n}$ for $n \geqslant 0$, and $\mathbb{Q}(\frac{n}{2}) = \mathbb{Q}(-\frac{1}{2})^{\otimes^{-n}}$ for $n < 0$. For any complex analytic space X with structure morphism $\pi : X \to \mathrm{pt}$, and any $M^\bullet \in D^b \operatorname{MHM}(X; T_s, N)$, we define the $\frac{n}{2}$ *twist of* $M^\bullet$ to be $M^\bullet(\frac{n}{2}) = M^\bullet \overset{L}{\otimes} \pi^*\big(\mathbb{Q}(\frac{n}{2})\big)$. This is consistent with the notation $M^\bullet(n)$ in Example 2.21.

Let V be a complex manifold, $f : V \to \mathbb{C}$ a holomorphic function, and $X = \operatorname{Crit}(f)$ its critical locus, as a complex analytic subspace of V. The perverse sheaf of vanishing cycles $\mathcal{PV}_{V,f}^\bullet \in \operatorname{Perv}(X)$ from §2.5 has a lift to a mixed Hodge module $\mathcal{HV}_{V,f}^\bullet$ in $\operatorname{MHM}(X; T_s, N)$, defined for each $c \in f(X)$ by

$$\mathcal{HV}_{V,f}^\bullet|_{X_c} = \phi_{f-c}^{pH}\big(\mathbb{Q}_V^H[\dim V]\big)\big|_{X_c}\big(\tfrac{1}{2}\dim V\big) \in \operatorname{MHM}(X_c; T_s, N). \tag{2.15}$$

Here the twist $(\frac{1}{2}\dim V)$ in (2.15), using the notation of Example 2.22, is included for the same reason as the $(-1)^{\dim V}$ in the definition (2.7) of $\tau_{V,f}$. It makes $\mathcal{HV}^\bullet_{V,f}$ act naturally under transformations which change dimension — without it, equations (5.7) and (6.15) below would have to include twists $(\frac{1}{2}n)$ for $n = \dim W - \dim V$. Then $\mathcal{HV}^\bullet_{V,f}$, $T_s : \mathcal{HV}^\bullet_{V,f} \to \mathcal{HV}^\bullet_{V,f}$ and $N : \mathcal{HV}^\bullet_{V,f} \to \mathcal{HV}^\bullet_{V,f}(-1)$ are related to $\mathcal{PV}^\bullet_{V,f}$ and $\tau_{V,f} : \mathcal{PV}^\bullet_{V,f} \to \mathcal{PV}^\bullet_{V,f}$ in §2.5 by

$$\mathcal{PV}^\bullet_{V,f} = \mathbf{rat}\big(\mathcal{HV}^\bullet_{V,f}\big) \quad \text{and} \quad \tau_{V,f} = \mathbf{rat}(T_s) \circ \exp\big(\mathbf{rat}(N)\big).$$

There is a Thom–Sebastiani Theorem for mixed Hodge modules due to Saito [25, Th. 5.4]. Applied to $\mathcal{HV}^\bullet_{V,f}$, it yields an analogue of Theorem 2.15:

Theorem 2.23. *Let V, W be complex manifolds and $f : V \to \mathbb{C}$, $g : W \to \mathbb{C}$ be holomorphic, so that $f \boxplus g : V \times W \to \mathbb{C}$ is holomorphic with $(f \boxplus g)(v, w) := f(v) + g(w)$. Set $X = \mathrm{Crit}(f)$ and $Y = \mathrm{Crit}(g)$ as complex analytic subspaces of V, W, so that $\mathrm{Crit}(f \boxplus g) = X \times Y$. Then there is a natural isomorphism*

$$\mathcal{HV}^\bullet_{V\times W, f\boxplus g} \cong \mathcal{HV}^\bullet_{V,f} \overset{L}{\boxtimes} \mathcal{HV}^\bullet_{W,g} \quad \textit{in } \mathrm{MHM}(X \times Y; T_s, N). \tag{2.16}$$

Note that this includes the analogue of $\tau_{V\times W, f\boxplus g} \cong \tau_{V,f} \overset{L}{\boxtimes} \tau_{W,g}$ in Theorem 2.15, as (2.16) holds in $\mathrm{MHM}(X \times Y; T_s, N)$ rather than just $\mathrm{MHM}(X \times Y)$. In this paper we will only ever apply Theorem 2.23 when $W = \mathbb{C}^n$, $g = z_1^2 + \cdots + z_n^2$ and $Y = \{0\}$. Combining (2.14) and (2.15) shows that

$$\mathcal{HV}^\bullet_{\mathbb{C},z^2} = \big(\mathbb{Q}(-\tfrac{1}{2})\big)(\tfrac{1}{2}) \cong \mathbb{Q}(0) \cong \mathbb{Q}^H_{\{0\}}.$$

Thus, by Theorem 2.23, $\mathbb{Q}^H_{\{0\}} \overset{L}{\boxtimes} \mathbb{Q}^H_{\{0\}} \cong \mathbb{Q}^H_{\{0\}}$, and induction on n, we see that

$$\mathcal{HV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \cong \mathbb{Q}^H_{\{0\}}. \tag{2.17}$$

As for (2.8), this isomorphism is natural up to sign, depending on a choice of orientation for the complex Euclidean space $(\mathbb{C}^n, \mathrm{d}z_1^2 + \cdots + \mathrm{d}z_n^2)$.

In the situation of §2.6, given two holomorphic functions $f, g : V \to \mathbb{C}$ and a local biholomorphism $\Phi : V \to V$ defined near $X = \mathrm{Crit}(f)$ in V with $g \circ \Phi = f$ and $\Phi|_X = \mathrm{id}_X$, we have an induced map $\Phi^H_* : \mathcal{HV}^\bullet_{V,f} \to \mathcal{HV}^\bullet_{V,g}$, defined by an analogous diagram to (2.10).

3 Action of symmetries on vanishing cycles

Here is our first main result, which answers Question 1.1(a).

Theorem 3.1. *Let V be a complex manifold, $f : V \to \mathbb{C}$ be holomorphic, and $X = \mathrm{Crit}(f)$, as a complex analytic subspace of V. Suppose $\Phi : V \to V$ is a local biholomorphism defined near X with $\Phi|_X = \mathrm{id}_X$ and $f \circ \Phi = f$. Then:*

(a) *As Φ is a local biholomorphism, $\mathrm{d}\Phi : TV \to \Phi^*(TV)$ is an isomorphism of vector bundles. Restricting to X gives*

$$\mathrm{d}\Phi|_X : TV|_X \longrightarrow \Phi|_X^*(TV) = \mathrm{id}_X^*(TV) = TV|_X,$$

an automorphism of the vector bundle $TV|_X$ on X. Hence $\det\bigl(\mathrm{d}\Phi|_X\bigr) : X \to \mathbb{C}\setminus\{0\}$ is a holomorphic map of complex analytic spaces.

Then $\det\bigl(\mathrm{d}\Phi|_X\bigr)$ maps $X \to \{\pm 1\} \subset \mathbb{C}\setminus\{0\}$, and is locally constant.

(b) *The isomorphism $\Phi_* : \mathcal{PV}^\bullet_{V,f} \to \mathcal{PV}^\bullet_{V,f}$ in Definition 2.17 is multiplication by $\det\bigl(\mathrm{d}\Phi|_X\bigr) : X \to \{\pm 1\}$ from* **(a)**.

(c) *The isomorphism $\Phi_*^H : \mathcal{HV}^\bullet_{V,f} \to \mathcal{HV}^\bullet_{V,f}$ of the mixed Hodge module of vanishing cycles defined at the end of §2.7 is multiplication by $\det\bigl(\mathrm{d}\Phi|_X\bigr) : X \to \{\pm 1\}$ from* **(a)**.

Here when we say $\Phi : V \to V$ *is a local biholomorphism defined near* X, we mean there is an open neighbourhood U of X in V and a holomorphic map $\Phi : U \to V$ with $U' = \Phi(U)$ open in V and $\Phi : U \to U'$ a biholomorphism, that is, a holomorphic map with holomorphic inverse. We generally leave the domain U of Φ implicit, for brevity. We prove parts (a) and (b)–(c) in §3.1–§3.2.

3.1 Part (a): $\det(\mathrm{d}\Phi|_X) = \pm 1$

We work in the situation of Theorem 3.1. For $x \in X \subseteq V$, we have an exact sequence of vector spaces

$$0 \longrightarrow T_x X \longrightarrow T_x V \xrightarrow{\ \mathrm{Hess}_x f\ } T_x^* V \longrightarrow T_x^* X \longrightarrow 0, \tag{3.1}$$

where $T_x X$ is the Zariski tangent space of X as a complex analytic space, and $\mathrm{Hess}_x f = (\partial^2 f)|_x$ is the Hessian of f at x. The action of Φ on V near x induces an automorphism of (3.1), giving a diagram:

$$\begin{tikzcd}
0 \arrow[r] & T_x X \arrow[r] \arrow[d, "{\mathrm{d}(\Phi|_X)|_x = \mathrm{id}_{T_x X}}"'] & T_x V \arrow[r, "\mathrm{Hess}_x f"'] \arrow[d, "\mathrm{d}\Phi|_x"'] & T_x^* V \arrow[r] \arrow[d, "(\mathrm{d}\Phi|_x^{-1})^*"] & T_x^* X \arrow[r] \arrow[d, "{(\mathrm{d}(\Phi|_X)|_x^{-1})^* = \mathrm{id}_{T_x^* X}}"] & 0 \\
0 \arrow[r] & T_x X \arrow[r] & T_x V \arrow[r, "\mathrm{Hess}_x f"] & T_x^* V \arrow[r] & T_x^* X \arrow[r] & 0.
\end{tikzcd} \tag{3.2}$$

Here the outer columns are identities as $\Phi|_X = \mathrm{id}_X$, and the outer squares obviously commute. We can show the central square commutes by taking second derivatives of $f \circ \Phi = f$ to get $\partial^2 f|_x \circ (\mathrm{d}\Phi|_x \otimes \mathrm{d}\Phi|_x) = \partial^2 f|_x$, and composing with $\mathrm{id} \otimes \mathrm{d}\Phi|_x^{-1}$. Thus (3.2) is commutative.

By elementary linear algebra, one can show that if $(E^\bullet, \mathrm{d})$ is a finite exact sequence of finite-dimensional vector spaces and $\phi^\bullet : E^\bullet \to E^\bullet$ is a chain map then $\prod_{k \text{ odd}} \det \phi^k = \prod_{k \text{ even}} \det \phi^k$. Applying this to (3.2) shows that

$$\det\bigl(\mathrm{id}_{T_x X}\bigr)\, \det\bigl((\mathrm{d}\Phi|_x^{-1})^*\bigr) = \det\bigl(\mathrm{d}\Phi|_x\bigr)\, \det\bigl(\mathrm{id}_{T_x^* X}\bigr). \tag{3.3}$$

But $\det\bigl((\mathrm{d}\Phi|_x^{-1})^*\bigr)=\det\bigl(\mathrm{d}\Phi|_x^{-1}\bigr)=\det\bigl(\mathrm{d}\Phi|_x\bigr)^{-1}$ and $\det\bigl(\mathrm{id}_{T_xX}\bigr)=\det\bigl(\mathrm{id}_{T_x^*X}\bigr)=1$, so (3.3) gives $\det\bigl(\mathrm{d}\Phi|_x\bigr)=\det\bigl(\mathrm{d}\Phi|_x\bigr)^{-1}$, so that $\det\bigl(\mathrm{d}\Phi|_x\bigr)=\pm 1$.

The same argument works with (analytic) coherent sheaves on X. The analogue of (3.1) is an exact sequence of coherent sheaves on X:

$$0 \longrightarrow TX \longrightarrow TV|_X \xrightarrow{(\mathrm{Hess}\, f)|_X} T^*V|_X \longrightarrow T^*X \longrightarrow 0,$$

where $TX := (T^*X)^\vee$ and T^*X are the tangent and cotangent sheaves of the complex analytic space X, and the analogue of (3.2) is the commutative diagram of coherent sheaves

$$\begin{tikzcd}
0 \arrow[r] & TX \arrow[r] \arrow[d, "{\mathrm{d}(\Phi|_X) \atop =\mathrm{id}_{TX}}"'] & TV|_X \arrow[r, "(\mathrm{Hess}\, f)|_X"'] \arrow[d, "\mathrm{d}\Phi|_X"'] & T^*V|_X \arrow[r] \arrow[d, "(\mathrm{d}\Phi|_X^{-1})^*"] & T^*X \arrow[r] \arrow[d, "{(\mathrm{d}(\Phi|_X)^{-1})^* \atop =\mathrm{id}_{T^*X}}"] & 0 \\
0 \arrow[r] & TX \arrow[r] & TV|_X \arrow[r, "(\mathrm{Hess}\, f)|_X"] & T^*V|_X \arrow[r] & T^*X \arrow[r] & 0.
\end{tikzcd} \tag{3.4}$$

We can make sense of the determinants of the columns in (3.4) using the theory of *determinant line bundles* of coherent sheaves, introduced by Knudsen and Mumford [13]. The main points are these:

(i) to any coherent sheaf $\mathcal{E}$ on X we can associate a determinant line bundle $\det\mathcal{E}$, a holomorphic line bundle on X. If $\mathcal{E}$ is (locally) equivalent in $D^b\operatorname{coh}(X)$ to a bounded complex $(\mathcal{F}^\bullet,\mathrm{d})$ with $\mathcal{F}^\bullet$ locally free, then there is a (local) canonical isomorphism $\det\mathcal{E}\cong\bigotimes_{k\in\mathbb{Z}}(\Lambda^{\mathrm{top}}\mathcal{F}^k)^{(-1)^k}$.

(ii) If $0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0$ is exact in $\operatorname{coh}(X)$, there is a canonical isomorphism $\det\mathcal{F}\cong\det\mathcal{E}\otimes\det\mathcal{G}$.

(iii) If $\alpha:\mathcal{E}\to\mathcal{F}$ is an isomorphism in $\operatorname{coh}(X)$, there is a natural isomorphism $\det\alpha:\det\mathcal{E}\to\det\mathcal{F}$. These are functorial, that is, if $\beta:\mathcal{F}\to\mathcal{G}$ is another isomorphism then $\det(\beta\circ\alpha)=(\det\beta)\circ(\det\alpha)$.

    If $\alpha:\mathcal{E}\to\mathcal{E}$ is an isomorphism then $\det\alpha:\det\mathcal{E}\to\det\mathcal{E}$ is an automorphism of the line bundle $\det\mathcal{E}$, so it is multiplication by a holomorphic function $X\to\mathbb{C}\setminus\{0\}$, which we will also refer to as $\det\alpha$.

The elementary linear algebra used above also holds for coherent sheaves. So from (3.4) we deduce the analogue of (3.3), in functions $X\to\mathbb{C}\setminus\{0\}$:

$$\det\bigl(\mathrm{id}_{TX}\bigr)\,\det\bigl((\mathrm{d}\Phi|_X^{-1})^*\bigr)=\det\bigl(\mathrm{d}\Phi|_X\bigr)\,\det\bigl(\mathrm{id}_{T^*X}\bigr).$$

As before we deduce that $\det\bigl(\mathrm{d}\Phi|_X\bigr)=\det\bigl(\mathrm{d}\Phi|_X\bigr)^{-1}$, which implies that as a morphism of complex analytic spaces, $\det\bigl(\mathrm{d}\Phi|_X\bigr)$ maps to $\{\pm1\}$ in $\mathbb{C}\setminus\{0\}$, and is locally constant. This proves Theorem 3.1(a).

Remark 3.2. The above proof is also valid in the algebraic rather than the analytic context, for schemes over a field $\mathbb{K}$, in the Zariski topology.

3.2 Parts (b)–(c): Φ_* is multiplication by $\det(\mathrm{d}\Phi|_X)$

Proposition 3.3. *Let V be a complex manifold, $f : V \to \mathbb{C}$ be holomorphic, $X = \mathrm{Crit}(f)$ as a complex analytic subspace of V, and $x \in X$. Suppose $\Phi : V \to V$ is a local biholomorphism defined near x with $\Phi|_X = \mathrm{id}_X$ and $f \circ \Phi = f$. Then $\det\bigl(\mathrm{d}\Phi|_x\bigr) = \pm 1$ as in §3.1, and $(\Phi|_{MF_f(x)})_* : \tilde{H}^*(MF_f(x), \mathbb{Q}) \to \tilde{H}^*(MF_f(x), \mathbb{Q})$ in Definition 2.10 is multiplication by $\det\bigl(\mathrm{d}\Phi|_x\bigr)$.*

Assuming Proposition 3.3, we will prove Theorem 3.1(b)–(c). In the situation of Theorem 3.1, define morphisms $\alpha, \beta : \mathcal{PV}^\bullet_{V,f} \to \mathcal{PV}^\bullet_{V,f}$ in $\mathrm{Perv}(X)$ by $\alpha = \Phi_*$ and β is multiplication by $\det\bigl(\mathrm{d}\Phi|_X\bigr) : X \to \{\pm 1\}$. For each $x \in X$ and $m \in \mathbb{Z}$, we have isomorphisms

$$\mathcal{H}^m(\alpha)_x, \mathcal{H}^m(\beta)_x : \mathcal{H}^m\bigl(\mathcal{PV}^\bullet_{V,f}\bigr)_x \longrightarrow \mathcal{H}^m\bigl(\mathcal{PV}^\bullet_{V,f}\bigr)_x.$$

Combining Proposition 3.3, the commutativity of (2.11), and $\alpha = \Phi_*$, shows that $\mathcal{H}^m(\alpha)_x$ is multiplication by $\det\bigl(\mathrm{d}\Phi|_x\bigr)$. But $\mathcal{H}^m(\beta)_x$ is also multiplication by $\det\bigl(\mathrm{d}\Phi|_x\bigr)$. So $\mathcal{H}^m(\alpha)_x = \mathcal{H}^m(\beta)_x$ for all x, m, and Corollary 2.6 shows that $\alpha = \beta$, proving Theorem 3.1(b).

Now Theorem 3.1(c) simply follows from the fact that the map Φ_*^H has underlying map Φ_* on the perverse sheaf level, that is, $\mathbf{rat}\bigl(\Phi_*^H\bigr) = \Phi_*$. Since the forgetful functor **rat** is faithful, as in (2.13), $\Phi_* = \det\bigl(\mathrm{d}\Phi|_x\bigr)\cdot$ implies that $\Phi_*^H = \det\bigl(\mathrm{d}\Phi|_x\bigr)\cdot$, as we have to prove.

It remains to prove Proposition 3.3. We begin with the following lemma:

Lemma 3.4. *Let V be a complex manifold, $f : V \to \mathbb{C}$ be holomorphic, and $X = \mathrm{Crit}(f)$ as a complex analytic subspace of V. Suppose $\Phi : V \to V$ is a local biholomorphism defined near $x \in V$ with $\Phi(x) = x$, and $\mathrm{d}\Phi|_{T_xV} = \mathrm{id}_{T_xV} : T_xV \to T_xV$, and $f \circ \Phi = f$. Then $\Phi|_{MF_f(x)} : MF_f(x) \to MF_f(x)$ in Definition 2.10 is isotopic to the identity, so $(\Phi|_{MF_f(x)})_* : \tilde{H}^*(MF_f(x), \mathbb{Q}) \to \tilde{H}^*(MF_f(x), \mathbb{Q})$ is also the identity.*

Proof. Informally, the lemma holds because $MF_f(x)$ is basically a submanifold of T_xV, so $\mathrm{d}\Phi|_{T_xV} = \mathrm{id}_{T_xV}$ implies that $\Phi|_{MF_f(x)}$ acts as the identity on $MF_f(x)$, up to isotopy.

More formally, choose a metric d on V and $0 < \epsilon \ll \delta \ll \delta' \ll 1$. Then we have two different models for the Milnor fibre $MF_f(x)$:

$$\begin{aligned} MF_f(x)^{d,\epsilon,\delta} &= \bigl\{v \in V : d(x,v) < \delta,\ f(v) - f(x) = \epsilon\bigr\}, \\ MF_f(x)^{d,\epsilon,\delta'} &= \bigl\{v \in V : d(x,v) < \delta',\ f(v) - f(x) = \epsilon\bigr\}. \end{aligned}$$

Since $\Phi(x) = x$ and $\delta \ll \delta'$ we have $\Phi\bigl(B_\delta(x)\bigr) \subseteq B_{\delta'}(x)$, so as $f \circ \Phi = f$ we see that $\Phi\bigl(MF_f(x)^{d,\epsilon,\delta}\bigr) \subseteq MF_f(x)^{d,\epsilon,\delta'}$. Also $MF_f(x)^{d,\epsilon,\delta} \subseteq MF_f(x)^{d,\epsilon,\delta'}$ as $\delta \ll \delta'$. Hence $\Phi|_{MF_f(x)^{d,\epsilon,\delta}}$ and $\mathrm{id}_{MF_f(x)^{d,\epsilon,\delta}}$ are both smooth maps $MF_f(x)^{d,\epsilon,\delta} \to MF_f(x)^{d,\epsilon,\delta'}$. We will show they are isotopic amongst such smooth maps.

Identify V near x with T_xV near 0, so that we treat points of V as vectors, and choose a Euclidean metric on T_xV, so that we can turn the 1-form $\mathrm{d}f$

into a vector field ∇f. For $G : MF_f(x)^{d,\epsilon,\delta} \times [0,1] \to \mathbb{R}$ smooth, define $F : MF_f(x)^{d,\epsilon,\delta} \times [0,1] \to V$ by $F(v) = (1-t)\Phi(v) + tv + G(v,t)\,(\nabla f)(v)$.

Using local estimates for $f, \nabla f, \nabla^2 f, \Phi, \nabla\Phi, \nabla^2\Phi$, we can show that provided ϵ, δ are chosen with $0 < \epsilon \ll \delta \ll 1$, there is a unique G, given approximately by $G(v,t) \approx [\epsilon + f(x) - f((1-t)\Phi(v) + tv)]/|(\nabla f)(v)|^2$, satisfying $G(v,0) = G(v,1) = 0$ for all v, for which $f \circ F = \epsilon + f(x)$. If also $\delta \ll \delta' \ll 1$ then F maps to $MF_f(x)^{d,\epsilon,\delta'} \subset V$. Therefore $F : MF_f(x)^{d,\epsilon,\delta} \times [0,1] \to MF_f(x)^{d,\epsilon,\delta'}$ is smooth with $F(v,0) = \Phi(v)$ and $F(v,1) = v$ for all $v \in MF_f(x)^{d,\epsilon,\delta}$. Thus, F is an isotopy from $\Phi|_{MF_f(x)^{d,\epsilon,\delta}}$ to $\mathrm{id}_{MF_f(x)^{d,\epsilon,\delta}}$ through smooth maps $MF_f(x)^{d,\epsilon,\delta} \to MF_f(x)^{d,\epsilon,\delta'}$, and $\Phi|_{MF_f(x)} : MF_f(x) \to MF_f(x)$ is isotopic to the identity. $\square$

This implies that in the situation of Proposition 3.3, considering V, f, x as fixed but $\Phi : V \to V$ as varying, the action $(\Phi|_{MF_f(x)})_* : \tilde H^*(MF_f(x), \mathbb{Q}) \to \tilde H^*(MF_f(x), \mathbb{Q})$ depends only Φ only via $\mathrm{d}\Phi|_x : T_xX \to T_xX$, since if Φ' is an alternative choice with $\mathrm{d}\Phi'|_x = \mathrm{d}\Phi|_x$ then

$$\begin{aligned}\big(\Phi'|_{MF_f(x)}\big)_* &= \big(\Phi|_{MF_f(x)}\big)_* \circ \big((\Phi^{-1}\circ\Phi')|_{MF_f(x)}\big)_* = \big(\Phi|_{MF_f(x)}\big)_* \circ \mathrm{id}\\ &= \big(\Phi|_{MF_f(x)}\big)_* : \tilde H^*(MF_f(x), \mathbb{Q}) \longrightarrow \tilde H^*(MF_f(x), \mathbb{Q}),\end{aligned}$$

using functoriality of pushforwards in the first step and applying Lemma 3.4 to $\Phi^{-1} \circ \Phi'$ in the second, noting that $\mathrm{d}(\Phi^{-1} \circ \Phi')|_x = \mathrm{id}$ as $\mathrm{d}\Phi'|_x = \mathrm{d}\Phi|_x$.

Fix V, f and $x \in X$ as above. We will describe the subgroup of elements $\mathrm{Aut}(T_xX)$ of the form $\mathrm{d}\Phi|_x : T_xX \to T_xX$, for $\Phi : V \to V$ a local biholomorphism defined near x with $\Phi|_X = \mathrm{id}_X$ and $f \circ \Phi = f$. The Zariski tangent space T_xX is a vector subspace of T_xV. Choose a complementary vector space $(T_xX)^\perp$ with $T_xV = T_xX \oplus (T_xX)^\perp$, and let $T_x^*V = T_x^*X \oplus (T_x^*X)^\perp$ be the corresponding splitting of T_x^*V. Then

$$S^2(T_x^*V) = S^2T_x^*X \oplus S^2(T_x^*X)^\perp \oplus \big(T_x^*X \otimes (T_x^*X)^\perp\big). \tag{3.5}$$

Exactness of (3.1) implies that under the splitting (3.5) we may write

$$\mathrm{Hess}_x\, f = \partial^2 f|_x = (0, q_x, 0), \tag{3.6}$$

where $q_x \in S^2(T_x^*X)^\perp$ is a nondegenerate quadratic form on $(T_xX)^\perp$.

Now let $\Phi : V \to V$ be a local biholomorphism defined near x with $\Phi|_X = \mathrm{id}_X$ and $f \circ \Phi = f$. Then we have a commutative diagram with exact rows:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
0 \arrow[r] & T_xX \arrow[r, "\begin{pmatrix}\mathrm{id}_{T_xX}\\ 0\end{pmatrix}"'] \arrow[d, "(\mathrm{d}\Phi|_x)|_{T_xX}=\mathrm{id}_{T_xX}"] & \begin{array}{c}T_xV =\\ T_xX\oplus(T_xX)^\perp\end{array} \arrow[r, "\begin{pmatrix}0 & \mathrm{id}_{(T_xX)^\perp}\end{pmatrix}"'] \arrow[d, "\mathrm{d}\Phi|_x=\begin{pmatrix}\mathrm{id}_{T_xX} & A\\ 0 & B\end{pmatrix}"] & (T_xX)^\perp \arrow[r] \arrow[d, "B"] & 0 \\
0 \arrow[r] & T_xX \arrow[r, "\begin{pmatrix}\mathrm{id}_{T_xX}\\ 0\end{pmatrix}"] & \begin{array}{c}T_xV =\\ T_xX\oplus(T_xX)^\perp\end{array} \arrow[r, "\begin{pmatrix}0 & \mathrm{id}_{(T_xX)^\perp}\end{pmatrix}"] & (T_xX)^\perp \arrow[r] & 0,
\end{tikzcd}$$

Here $\mathrm{d}\Phi|_x$ preserves the subspace $T_xX \subseteq T_xV$ and is the identity there, as $\Phi|_X = \mathrm{id}_X$. Hence $\mathrm{d}\Phi|_x$ is of the form $\left(\begin{smallmatrix} \mathrm{id} & A \\ 0 & B \end{smallmatrix}\right)$ as shown, for linear $A : (T_xX)^\perp \to T_xX$ and $B : (T_xX)^\perp \to (T_xX)^\perp$. Also $\mathrm{d}\Phi|_x$ preserves $\mathrm{Hess}_x\, f$ on T_xV, as $f \circ \Phi = f$. So (3.6) implies that B preserves the nondegenerate quadratic form q_x on $(T_xX)^\perp$. That is, B is a complex orthogonal matrix in $\mathrm{O}\big((T_xX)^\perp, q_x\big)$.

In the next part of the proof, with V, f, x fixed, we will construct explicit examples of Φ with $\mathrm{d}\Phi|_x = \left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & B \end{smallmatrix}\right)$ for any $B \in \mathrm{O}\big((T_xX)^\perp, q_x\big)$. Locally near $x \in V$, choose a splitting $T^*V = E \oplus F$ for E, F holomorphic vector bundles with $E|_x = T_x^*X$ and $F|_x = (T_x^*X)^\perp$. Then $\mathrm{d}f \in H^0(T^*V)$ splits as $\mathrm{d}f = \alpha \oplus \beta$ for $\alpha \in H^0(E)$ and $\beta \in H^0(F)$, and X is defined by $\alpha = \beta = 0$. Since $\mathrm{Hess}_x\, f = \partial(\mathrm{d}f)|_x$ is nondegenerate on $(T_xX)^\perp$, we see that $\nabla\beta|_x : T_xV \to F|_x$ induces an isomorphism $(T_xX)^\perp \to F|_x$, so $\nabla\beta|_x$ is surjective.

Therefore $\beta^{-1}(0)$ is a submanifold of V near x. So we can choose an open neighbourhood V' of x in V such that E, F, α, β are defined on V', and $X \cap V' = \{v \in V' : \alpha(v) = \beta(v) = 0\}$, and the complex analytic subspace $U = \{v \in V' : \beta(v) = 0\}$ is a submanifold of V'. Set $e = f|_U : U \to \mathbb{C}$. Then the natural isomorphism $T^*U \cong E|_U$ identifies $\mathrm{d}e \in H^0(T^*U)$ with $\alpha|_U \in H^0(E|_U)$.

Now $X \cap V'$ is the complex analytic subspace of V' defined by $\alpha = \beta = 0$, as $\mathrm{d}f = \alpha \oplus \beta$. Also $\mathrm{Crit}(e)$ is the complex analytic subspace of U defined by $\alpha = 0$, and U is the complex submanifold of V' defined by $\beta = 0$. Hence $\mathrm{Crit}(e) = X \cap V'$ as complex analytic subspaces of V'. The proof of Theorem 5.1(i) in §5.1 below therefore shows that there exists a local biholomorphism $\Psi : V \to U \times \mathbb{C}^n$ defined near $x \in V$, where $n = \dim V - \dim U$ with $\Psi : u \mapsto (u, 0)$ for $u \in U$, and if $\Psi(v) = \big(u, (z_1, \ldots, z_n)\big)$ then $f(v) = e(u) + z_1^2 + \cdots + z_n^2$. That is, Ψ identifies $f : V \to \mathbb{C}$ with $e \boxplus z_1^2 + \cdots + z_n^2 : U \times \mathbb{C}^n \to \mathbb{C}$. Also,

$$\mathrm{d}\Psi|_x : T_xV = T_xX \oplus (T_xX)^\perp \longrightarrow T_{(x,0)}(U \times \mathbb{C}^n) = T_xU \oplus T_0\mathbb{C}^n$$

identifies $T_xX \cong T_xU$ and $(T_xX)^\perp \cong T_0\mathbb{C}^n \cong \mathbb{C}^n$ and $q_x \in S^2(T_x^*X)^\perp$ with $\mathrm{d}z_1^2 + \cdots + \mathrm{d}z_n^2$ in $S^2(T_0^*\mathbb{C}^n)$.

Let $B \in \mathrm{O}\big((T_xX)^\perp, q_x\big)$. Then the isomorphism $(T_xX)^\perp \cong \mathbb{C}^n$ identifies B with a unique orthogonal matrix $C \in \mathrm{O}(n, \mathbb{C})$. Define a local biholomorphism $\Phi : V \to V$ near x by $\Phi = \Psi^{-1} \circ (\mathrm{id}_U \times C) \circ \Psi$. Then $\Phi|_X = \mathrm{id}_X$ as $\Psi(X) \subseteq U \times \{0\}$ and $\mathrm{id}_U \times C|_{U \times \{0\}} = \mathrm{id}_{U \times \{0\}}$, and

$$\begin{aligned} f \circ \Phi &= f \circ \Psi^{-1} \circ (\mathrm{id}_U \times C) \circ \Psi = (e \boxplus z_1^2 + \cdots + z_n^2) \circ (\mathrm{id}_U \times C) \circ \Psi \\ &= (e \boxplus z_1^2 + \cdots + z_n^2) \circ \Psi = f. \end{aligned}$$

On reduced cohomology of Milnor fibres we have a commutative diagram:

$$
\begin{tikzcd}
\tilde{H}^*\bigl(MF_f(x);\mathbb{Q}\bigr) \arrow[r, "(\Phi|_{MF_f(x)})_*"'] \arrow[d, "(\Psi|_{MF_f(x)})_*"] & \tilde{H}^*\bigl(MF_f(x);\mathbb{Q}\bigr) \\
\tilde{H}^*\bigl(MF_{e\boxplus z_1^2+\cdots+z_n^2}(x,0);\mathbb{Q}\bigr) \arrow[d, "(2.1)"] & \tilde{H}^*\bigl(MF_{e\boxplus z_1^2+\cdots+z_n^2}(x,0);\mathbb{Q}\bigr) \arrow[u, "(\Psi^{-1}|_{MF_{e\boxplus z_1^2+\cdots+z_n^2}(x,0)})_*"'] \\
\tilde{H}^*\bigl(MF_e(x);\mathbb{Q}\bigr)\otimes_{\mathbb{Q}} \tilde{H}^*\bigl(MF_{z_1^2+\cdots+z_n^2}(0);\mathbb{Q}\bigr) \arrow[r, "(\mathrm{id}_U\,|_{MF_e(x)})_*\otimes (C|_{MF_{z_1^2+\cdots+z_n^2}(0)})_*"] & \tilde{H}^*\bigl(MF_e(x);\mathbb{Q}\bigr)\otimes_{\mathbb{Q}} \tilde{H}^*\bigl(MF_{z_1^2+\cdots+z_n^2}(0);\mathbb{Q}\bigr), \arrow[u, "(2.1)"']
\end{tikzcd}
\tag{3.7}
$$

where the isomorphisms (2.1) are as in Theorem 2.9. On the bottom line, $(\mathrm{id}_U\,|_{MF_e(x)})_* = \mathrm{id}$, and $(C|_{MF_{z_1^2+\cdots+z_n^2}(0)})_*$ is multiplication by $\det C = \pm 1$ on $\tilde{H}^*\bigl(MF_{z_1^2+\cdots+z_n^2}(0);\mathbb{Q}\bigr) \cong \mathbb{Q}$, as in Example 2.18. Hence $(\Phi|_{MF_f(x)})_*$ in (3.7) is multiplication by $\det C$. But $\mathrm{d}\Phi|_x = \bigl(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & B \end{smallmatrix}\bigr)$ and $\det B = \det C$, so $(\Phi|_{MF_f(x)})_*$ is multiplication by $\det\bigl(\mathrm{d}\Phi|_x\bigr)$.

We have now proved that for each choice of $(T_xX)^\perp \subseteq T_xV$ with $T_xV = T_xX \oplus (T_xX)^\perp$ and each $B \in \mathrm{O}\bigl((T_xX)^\perp, q_x\bigr)$, we can construct an example of a local biholomorphism $\Phi : V \to V$ defined near x with $\Phi|_X = \mathrm{id}_X$ and $f \circ \Phi = f$, such that $\mathrm{d}\Phi|_x = \bigl(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & B \end{smallmatrix}\bigr)$ in the splitting $T_xV = T_xX \oplus (T_xX)^\perp$, and $(\Phi|_{MF_f(x)})_* : \tilde{H}^*(MF_f(x),\mathbb{Q}) \to \tilde{H}^*(MF_f(x),\mathbb{Q})$ is multiplication by $\det\bigl(\mathrm{d}\Phi|_x\bigr)$, as in Proposition 3.3. Lemma 3.4 now implies that $(\Phi|_{MF_f(x)})_* = \det\bigl(\mathrm{d}\Phi|_x\bigr)\cdot$ for any such Φ with $\mathrm{d}\Phi|_x = \bigl(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & B \end{smallmatrix}\bigr)$.

As above, the allowed values of $\mathrm{d}\Phi|_x : T_xX \to T_xX$, which form a group, are of the form $\mathrm{d}\Phi|_x = \bigl(\begin{smallmatrix} \mathrm{id} & A \\ 0 & B \end{smallmatrix}\bigr)$ for B orthogonal. So far, our proof shows that $(\Phi|_{MF_f(x)})_* = \det\bigl(\mathrm{d}\Phi|_x\bigr)\cdot$ provided $A = 0$. But the representation of $\mathrm{d}\Phi|_x$ as $\bigl(\begin{smallmatrix} \mathrm{id} & A \\ 0 & B \end{smallmatrix}\bigr)$ depends on the choice of $(T_xX)^\perp$, and different choices of $(T_xX)^\perp$ give different families of transformations $\mathrm{d}\Phi|_x$ with $A = 0$.

In fact, the subset of $\mathrm{d}\Phi|_x$ which are of the form $\bigl(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & B \end{smallmatrix}\bigr)$ for some choice of $(T_xX)^\perp$ generates the full group of all $\mathrm{d}\Phi|_x = \bigl(\begin{smallmatrix} \mathrm{id} & A \\ 0 & B \end{smallmatrix}\bigr)$. Since $\mathrm{d}\Phi|_x \mapsto (\Phi|_{MF_f(x)})_*$ and $\mathrm{d}\Phi|_x \mapsto \det\bigl(\mathrm{d}\Phi|_x\bigr)$ are both group morphisms, and they are equal on a generating subset for the group of $\mathrm{d}\Phi|_x$, they are equal. This proves Proposition 3.3, and hence Theorem 3.1(b),(c).

4 Dependence of $\mathcal{PV}^\bullet_{V,f}$ on f

Here is our second main result, answering Question 1.1(b).

Theorem 4.1. *Let V be a complex manifold, $f : V \to \mathbb{C}$ a holomorphic function, and X be open and closed in* $\mathrm{Crit}(f)$ *as a complex analytic subspace of V. Write I_X for the ideal of holomorphic functions vanishing on X, as a subsheaf of the complex analytic sheaf $\mathcal{O}_V$ of holomorphic functions on V, so that $I_X = (\mathrm{d}f)$ near X.*

Suppose $g : V \to \mathbb{C}$ is a second holomorphic function with

$$f + I_X^3 = g + I_X^3 \tag{4.1}$$

in $H^0(\mathcal{O}_V/I_X^3)$. Then X is also open and closed in $\mathrm{Crit}(g)$, *and there is a canonical isomorphism of the perverse sheaves of vanishing cycles*

$$\Lambda_{f,g} : \mathcal{PV}^\bullet_{V,f}|_X \longrightarrow \mathcal{PV}^\bullet_{V,g}|_X. \tag{4.2}$$

Also the following commutes, where $\tau_{V,f}, \tau_{V,g}$ are as in (2.7):

$$\begin{tikzcd} \mathcal{PV}^\bullet_{V,f} \arrow[rr, "\Lambda_{f,g}"'] \arrow[d, "\tau_{V,f}"] && \mathcal{PV}^\bullet_{V,g} \arrow[d, "\tau_{V,g}"'] \\ \mathcal{PV}^\bullet_{V,f} \arrow[rr, "\Lambda_{f,g}"] && \mathcal{PV}^\bullet_{V,g}. \end{tikzcd} \tag{4.3}$$

If $h : V \to \mathbb{C}$ is a third holomorphic function with

$$f + I_X^3 = g + I_X^3 = h + I_X^3 \tag{4.4}$$

then $\Lambda_{f,h} = \Lambda_{g,h} \circ \Lambda_{f,g}$. Also $\Lambda_{g,f} = \Lambda_{f,g}^{-1}$, and $\Lambda_{f,f}$ is the identity morphism.

All these statements hold also on the Hodge module level: under the same assumptions on $f, g : V \to \mathbb{C}$, we have a canonical isomorphism of mixed Hodge modules of vanishing cycles

$$\Lambda^H_{f,g} : \mathcal{HV}^\bullet_{V,f}|_X \longrightarrow \mathcal{HV}^\bullet_{V,g}|_X \tag{4.5}$$

in $\mathrm{MHM}(X; T_s, N)$, *and given a third function h on V as above, the isomorphisms satisfy the cocycle condition $\Lambda^H_{f,h} = \Lambda^H_{g,h} \circ \Lambda^H_{f,g}$.*

Remark 4.2. Theorem 4.1 allows us to define perverse sheaves and mixed Hodge modules of vanishing cycles $\mathcal{PV}^\bullet_{\hat V,\hat f}, \mathcal{HV}^\bullet_{\hat V,\hat f}$ for a class of formal functions or formal power series $\hat f$.

Let V be a complex manifold and X a closed complex analytic subspace of V, and write $\hat V$ for the formal completion of V along X. As a complex analytic scheme, $\hat V = (\hat V, \mathcal{O}_{\hat V})$ has topological space $\hat V = X$ and structure sheaf $\mathcal{O}_{\hat V} = \lim_{n\to\infty} \mathcal{O}_V/I_X^n$, where $I_X \subset \mathcal{O}_V$ is the sheaf of ideals associated to X. Let $\hat f$ be a formal function on $\hat V$, that is, $\hat f \in H^0(\mathcal{O}_{\hat V})$, and suppose $\mathrm{Crit}(\hat f) = X \subset \hat V$ as a (formal) complex analytic subspace of $\hat V$, that is, $I_{\mathrm{d}\hat f} = I_X \subset \mathcal{O}_{\hat V}$.

Choose open subsets $U_i \subseteq V$ and holomorphic functions $f_i : U_i \to \mathbb{C}$ with $f_i + I_X^3 = \hat f + I_X^3$ on $U_i \cap X$ for $i \in I$ such that $\{U_i \cap X : i \in I\}$ is an open cover of X. We have perverse sheaves of vanishing cycles $\mathcal{PV}^\bullet_{U_i,f_i}$ on $U_i \cap X$. For $i, j \in I$ we have $f_i + I_X^3 = \hat f + I_X^3 = f_j + I_X^3$ on $U_i \cap U_j \cap X$, and so Theorem 4.1 gives a canonical isomorphism $\mathcal{PV}^\bullet_{U_i,f_i}|_{U_i\cap U_j\cap X} \cong \mathcal{PV}^\bullet_{U_j,f_j}|_{U_i\cap U_j\cap X}$. These isomorphisms are compatible on triple overlaps $U_i \cap U_j \cap U_k \cap X$ by the second part of Theorem 4.1.

Thus, by Theorem 2.5(ii) there exists a perverse sheaf $\mathcal{PV}^\bullet_{\hat V,\hat f}$ on X, unique up to isomorphism, with $\mathcal{PV}^\bullet_{\hat V,\hat f}|_{U_i\cap X} \cong \mathcal{PV}^\bullet_{U_i,f_i}$ for each $i \in I$. The same

argument gives a mixed Hodge module $\mathcal{HV}^\bullet_{\hat V,\hat f}$. Note that we have not defined perverse sheaves or mixed Hodge modules on $\hat V$, nor functors $\phi^p_{\hat f},\phi^{pH}_{\hat f}$, but only the particular perverse sheaf $\mathcal{PV}^\bullet_{\hat V,\hat f}$ and mixed Hodge module $\mathcal{HV}^\bullet_{\hat V,\hat f}$ on X.

Defining vanishing cycles for formal functions $\hat f$ of this kind is a natural question in Donaldson–Thomas theory. Moduli schemes $\mathcal M$ of stable coherent sheaves on Calabi–Yau 3-folds may be written locally as critical loci $\mathrm{Crit}(\hat f)$ of formal functions $\hat f$, as in Kontsevich and Soibelman [14] and the authors [6].

We begin the proof of Theorem 4.1 with the following proposition. It is based on Moser's proof of Darboux' Theorem in symplectic geometry [19].

Proposition 4.3. *Let V,f,X,I_X and g be as in Theorem 4.1, and $x\in X$. Then there exist open neighbourhoods U_x,U_x' of x in V and a biholomorphism $\Psi_x:U_x\to U_x'$ such that $U_x\cap X=U_x'\cap X$, and $\Psi_x\vert_{U_x\cap X}=\mathrm{id}_{U_x\cap X}$, and $\mathrm{d}\Psi_x\vert_{U_x\cap X}=\mathrm{id}_{TV\vert_{U_x\cap X}}$, and $g\circ\Psi_x=f\vert_{U_x}$.*

Proof. Equation (4.1) shows that $f-g\in I_X^3=I_X^2\cdot I_X$. Now I_X is generated by $\mathrm{d}f$, and (4.1) implies that I_X is also generated by $\mathrm{d}\bigl((1-t)f+tg\bigr)$ near X for $t\in[0,1]$. Thus, for each $t\in[0,1]$ we can choose an open neighbourhood W of x in V and a holomorphic vector field $v_t\in H^0(I_X^2\cdot TV\vert_W)$ such that

$$\mathrm{d}\bigl((1-t)f+tg\bigr)\cdot v_t=f-g. \tag{4.6}$$

Using compactness of $[0,1]$ we can show that we can use the same open W for each $t\in[0,1]$, and we can also choose v_t to depend smoothly on $t\in[0,1]$.

We wish to choose an open neighbourhood U_x of x in W and a smooth family $\Psi_x^t:t\in[0,1]$ of holomorphic maps $\Psi_x^t:U_x\to W$, satisfying the o.d.e.

$$\frac{\mathrm{d}\Psi_x^t}{\mathrm{d}t}=(\Psi_x^t)^*(v_t)\quad\text{in } H^0\bigl((\Psi_x^t)^*(TV)\bigr) \tag{4.7}$$

with initial value $\Psi_x^0=\mathrm{id}_{U_x}:U_x\to W$.

This means that for each $u\in U_x$, if we define $\gamma_u:[0,1]\to W$ by $\gamma_u(t)=\Psi_x^t(u)$, then γ_u is a smooth curve in W with $\gamma_u(0)=u$ and

$$\frac{\mathrm{d}\gamma_u}{\mathrm{d}t}(t)=v_t\vert_{\gamma_u(t)}\quad\text{for } t\in[0,1]. \tag{4.8}$$

Standard results on ordinary differential equations show that solutions of (4.8) exist locally on $[0,1]$ (that is, exist on some $[0,\epsilon)$ for $0<\epsilon\leqslant 1$) and are unique with the initial condition $\gamma_u(0)=u$. However, as W is noncompact, for given $u\in W$ it might happen that a flow-line γ_u reaches the noncompact edge of W at $t=\epsilon\leqslant 1$, so that γ_u exists only on $[0,\epsilon)\subsetneq[0,1]$, and not on $[0,1]$.

By assumption $v_t\in H^0(I_X^2\cdot TV)$, so v_t is zero on X and at x, and is small near x uniformly for all $t\in[0,1]$. Thus, a flow-line γ_u starting at u near x at time $t=0$ initially moves slowly, and stays near x for a long time. Hence we can choose an open neighbourhood U_x of x in W so that solutions γ_u to (4.8) for $t\in[0,1]$ with $\gamma_u(0)=u$ exist for all $u\in U_x$. Then we set $\Psi_x^t(u)=\gamma_u(t)$ for

$u \in U_x$ and $t \in [0,1]$, and $\Psi_x^t : U_x \to W$ for $t \in [0,1]$ are a smooth family of holomorphic maps satisfying (4.7) and $\Psi_x^0 = \mathrm{id}_{U_x}$.

We now have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigl[\bigl((1-t)f+tg\bigr)\circ\Psi_x^t\bigr] \\
&\quad= \bigl[\tfrac{\mathrm{d}}{\mathrm{d}t}\bigl((1-t)f+tg\bigr)\bigr]\circ\Psi_x^t + (\Psi_x^t)^*\bigl(\mathrm{d}((1-t)f+tg)\bigr)\cdot\bigl(\tfrac{\mathrm{d}\Psi_x^t}{\mathrm{d}t}\bigr) \\
&\quad= (g-f)\circ\Psi_x^t + (\Psi_x^t)^*\bigl(\mathrm{d}((1-t)f+tg)\bigr)\cdot(\Psi_x^t)^*(v_t) \\
&\quad= (\Psi_x^t)^*(g-f) + (\Psi_x^t)^*\bigl[\mathrm{d}((1-t)f+tg)\cdot v_t\bigr] \\
&\quad= (\Psi_x^t)^*(g-f) + (\Psi_x^t)^*(f-g) = 0,
\end{aligned}$$

using (4.7) in the second step and (4.6) in the fourth. So $((1-t)f+tg)\circ\Psi_x^t$ is independent of t. At $t=0$ it is $f\circ\mathrm{id}_{U_x} = f|_{U_x}$, and at $t=1$ it is $g\circ\Psi_x^1$. Thus $g\circ\Psi_x^1 = f|_{U_x}$. Set $\Psi_x = \Psi_x^1$, so that $g\circ\Psi_x = f|_{U_x}$, as we have to prove. Also Ψ_x is a local biholomorphism, so making U_x smaller we can suppose $\Psi_x : U_x \to U_x' := \Psi_x(U_x)$ is a biholomorphism.

Since $v_t \in H^0(I_X^2\cdot TV|_W)$, and Ψ_x is the time-dependent exponential of v_t for $t\in[0,1]$, we see that $\Psi_x + I_X^2|_{U_x} = \mathrm{id} + I_X^2|_{U_x}$. So $\Psi_x + I_X|_{U_x} = \mathrm{id} + I_X|_{U_x}$, which is equivalent to $\Psi_x|_{U_x\cap X} = \mathrm{id}_{U_x\cap X}$, and differentiating $\Psi_x + I_X^2|_{U_x} = \mathrm{id} + I_X^2|_{U_x}$ gives $\mathrm{d}\Psi_x + I_X|_{U_x} = \mathrm{d}(\mathrm{id}) + I_X|_{U_x}$, which is equivalent to $\mathrm{d}\Psi_x|_{U_x\cap X} = \mathrm{id}_{TV|_{U_x\cap X}}$. This proves the proposition. □

Let V, f, X, I_X and g be as in Theorem 4.1. We will now construct the isomorphism $\Lambda_{f,g}$ in (4.2). For each $x \in X$, choose U_x, U_x', Ψ_x as in Proposition 4.3. Then Definition 2.17 defines an isomorphism in $\mathrm{Perv}(U_x\cap X)$:

$$(\Psi_x)_* : \bigl(\mathcal{PV}_{V,f}^\bullet\bigr)\big|_{U_x\cap X} \longrightarrow \bigl(\mathcal{PV}_{V,g}^\bullet\bigr)\big|_{U_x\cap X}. \tag{4.9}$$

Let $x, y \in X$. Then we have a biholomorphism

$$\Phi_{xy} := \Psi_y^{-1}\circ\Psi_x|_{\Psi_x^{-1}(U_y')} : \Psi_x^{-1}(U_y') \longrightarrow \Psi_y^{-1}(U_x'). \tag{4.10}$$

Here $\Psi_x^{-1}(U_y')\cap X = \Psi_y^{-1}(U_x') = U_x\cap U_y\cap X$, and $\Phi_{xy}|_{U_x\cap U_y\cap X} = \mathrm{id}_{U_x\cap U_y\cap X}$. As $\Psi_y\circ\Phi_{xy} = \Psi_x|_{\Psi_x^{-1}(U_y')}$, by functoriality of pushforwards Φ_* we have

$$\begin{aligned}
(\Psi_y)_*|_{U_x\cap U_y\cap X}\circ(\Phi_{xy})_* = (\Psi_x)_*|_{U_x\cap U_y\cap X} :& \\
\bigl(\mathcal{PV}_{V,f}^\bullet\bigr)\big|_{U_x\cap U_y\cap X} \longrightarrow \bigl(\mathcal{PV}_{V,g}^\bullet\bigr)\big|_{U_x\cap U_y\cap X}&.
\end{aligned} \tag{4.11}$$

From (4.10) we see that

$$\begin{gathered}
f\circ\Phi_{xy} = f\circ\Psi_y^{-1}\circ\Psi_x|_{\Psi_x^{-1}(U_y')} = g\circ\Psi_x|_{\Psi_x^{-1}(U_y')} = f|_{\Psi_x^{-1}(U_y')}, \\
\mathrm{d}\Phi_{xy}|_{U_x\cap U_y\cap X} = (\mathrm{d}\Psi_y|_{U_x\cap U_y\cap X})^{-1}\circ\mathrm{d}\Psi_x|_{U_x\cap U_y\cap X} = \mathrm{id}_{TV|_{U_x\cap U_y\cap X}}.
\end{gathered}$$

We can now apply Theorem 3.1 to $f : V \to \mathbb{C}$, and the local biholomorphism Φ_{xy}. Since $\mathrm{d}\Phi_{xy}|_{U_x\cap U_y\cap X}$ is the identity, $\det\bigl(\mathrm{d}\Phi_{xy}|_{U_x\cap U_y\cap X}\bigr) = 1$, so Theorem 3.1 says that $(\Phi_{xy})_*$ is the identity. Thus (4.11) becomes

$$(\Psi_y)_*|_{U_x\cap U_y\cap X} = (\Psi_x)_*|_{U_x\cap U_y\cap X} \quad \text{for all } x, y \in X. \tag{4.12}$$

We have constructed an open cover $\{U_x \cap X : x \in X\}$ of X, and isomorphisms of perverse sheaves (4.9) for each $U_x \cap X$ in the cover which agree on overlaps $(U_x \cap X) \cap (U_y \cap X)$ by (4.12). Therefore Theorem 2.5(i) gives a unique isomorphism $\Lambda_{f,g}$ as in (4.2) with $\Lambda_{f,g}|_{U_x \cap X} = (\Psi_x)_*$ for all $x \in X$.

We claim that $\Lambda_{f,g}$ is independent of the choices of U_x, U'_x, Ψ_x for $x \in X$. To see this, let $\tilde{U}_x, \tilde{U}'_x, \tilde{\Psi}_x$ be alternative choices, giving an isomorphism $\tilde{\Lambda}_{f,g}$. Then for $x \in X$, the proof of (4.11) shows that

$$\Lambda_{f,g}|_{U_x \cap \tilde{U}_x \cap X} = (\Psi_x)_*|_{U_x \cap \tilde{U}_x \cap X} = (\tilde{\Psi}_x)_*|_{U_x \cap \tilde{U}_x \cap X} = \tilde{\Lambda}_{f,g}|_{U_x \cap \tilde{U}_x \cap X}.$$

Since the $U_x \cap \tilde{U}_x \cap X$ for $x \in X$ are an open cover of X, this forces $\Lambda_{f,g} = \tilde{\Lambda}_{f,g}$ by Theorem 2.5(i).

To see that (4.3) commutes, note that pushforwards $(\Psi_x)_*$ in (4.9) satisfy

$$(\Psi_x)_* \circ \tau_{V,f}|_{U_x \cap X} = \tau_{V,g}|_{U_x \cap X} \circ (\Psi_x)_* : \bigl(\mathcal{PV}^\bullet_{V,f}\bigr)\big|_{U_x \cap X} \longrightarrow \bigl(\mathcal{PV}^\bullet_{V,g}\bigr)\big|_{U_x \cap X}.$$

Since $\Lambda_{f,g}|_{U_x \cap X} = (\Psi_x)_*$, the restriction of (4.3) to $U_x \cap X$ commutes. Thus (4.3) commutes by Theorem 2.5(i), as the $U_x \cap X$ for $x \in X$ cover X.

Now let h be a third holomorphic function satisfying (4.4). Construct $\Lambda_{f,g}$ using data U_x, U'_x, Ψ_x and $\Lambda_{g,h}$ using data $\tilde{U}_x, \tilde{U}'_x, \tilde{\Psi}_x$ for $x \in X$. Define $\hat{U}_x = \Psi_x^{-1}(\tilde{U}_x) \subseteq U_x$, $\hat{\Psi}_x = \tilde{\Psi}_x \circ \Psi_x|_{\hat{U}_x}$, and $\hat{U}'_x = \hat{\Psi}_x(\hat{U}_x)$. Then

$$h \circ \hat{\Psi}_x = h \circ \tilde{\Psi}_x \circ \Psi_x|_{\hat{U}_x} = g \circ \Psi_x|_{\hat{U}_x} = f|_{\hat{U}_x}.$$

Hence $\hat{U}_x, \hat{U}'_x, \hat{\Psi}_x$ for $x \in X$ are possible choices in the construction of $\Lambda_{f,h}$, which is independent of these choices. For $x \in X$ we have

$$\begin{aligned}\Lambda_{f,h}|_{\hat{U}_x \cap X} &= \bigl(\hat{\Psi}_x\bigr)_* = \bigl(\tilde{\Psi}_x \circ \Psi_x|_{\hat{U}_x}\bigr)_* = \bigl(\tilde{\Psi}_x\bigr)_*\big|_{\hat{U}_x \cap X} \circ \bigl(\Psi_x\bigr)_*\big|_{\hat{U}_x \cap X} \\ &= (\Lambda_{g,h})|_{\hat{U}_x \cap X} \circ (\Lambda_{f,g})|_{\hat{U}_x \cap X} = (\Lambda_{g,h} \circ \Lambda_{f,g})|_{\hat{U}_x \cap X},\end{aligned}$$

using functoriality of Φ_* in Definition 2.17. As the $\hat{U}_x \cap X$ for $x \in X$ are an open cover of X, this gives $\Lambda_{f,h} = \Lambda_{g,h} \circ \Lambda_{f,g}$ by Theorem 2.5(i), as we have to prove. Then $\Lambda_{f,f} = \mathrm{id}$ follows from $\Lambda_{f,g} = \Lambda_{f,g} \circ \Lambda_{f,f}$ and $\Lambda_{f,g}$ an isomorphism, and $\Lambda_{g,f} = \Lambda_{f,g}^{-1}$ follows from $\Lambda_{f,f} = \Lambda_{g,f} \circ \Lambda_{f,g}$.

To conclude, we need to discuss the case of Hodge modules. The existence of the maps $\Lambda^H_{f,g}$ follows as above from Proposition 4.3, functoriality of the vanishing cycle module under biholomorphisms, and the stack property of mixed Hodge modules, Theorem 2.19(viii). The fact that the $\Lambda^H_{f,g}$ are isomorphisms gluing correctly on overlaps follows as usual from faithfulness of the functor **rat**, and the corresponding results on the perverse sheaf level. Finally the monodromy actions also match by (4.3), so the gluing is correct in the category $\mathrm{MHM}(X; T_s, N)$. This completes the proof of Theorem 4.1.

5 Stabilizing vanishing cycles

In this section, we will give our answer to Question 1.1(c). To state the result, which is Theorem 5.2 below, we first need some geometric facts.

Theorem 5.1. *Let W be a complex manifold, $V \subseteq W$ a complex submanifold, and $g : W \to \mathbb{C}$ be holomorphic, so that $f := g|_V : V \to \mathbb{C}$ is also holomorphic. Then $X := \mathrm{Crit}(f)$ is a complex analytic subspace of V, and hence of W, and $Y := \mathrm{Crit}(g)$ is a complex analytic subspace of W. Suppose that $X = Y$, as complex analytic subspaces of W. Then:*

(i) *Let $x \in X \subseteq V \subseteq W$. Then there exists an open neighbourhood W' of x in W and a holomorphic map $\Psi : W' \to V \times \mathbb{C}^n$, which is a biholomorphism with an open neighbourhood of $(x, 0)$ in $V \times \mathbb{C}^n$, where $n = \dim W - \dim V$, such that if $v \in W' \cap V$ then $\Psi(v) = \bigl(v, (0, \ldots, 0)\bigr)$, and if $w \in W'$ with $\Psi(w) = \bigl(v, (z_1, \ldots, z_n)\bigr)$, then $g(w) = f(v) + z_1^2 + \cdots + z_n^2$.*

(ii) *Write ν for the normal bundle of V in W, so that $\nu|_X$ is a holomorphic vector bundle on the complex analytic space X of rank n. There exists a unique $q \in H^0(S^2\nu^*|_X)$ which is a nondegenerate, holomorphic quadratic form on $\nu|_X$, with the following property.*

Suppose x, W', Ψ are as in **(i)**, *and write $Y = W' \cap X$, as an open complex analytic subspace of X. Then we have a commutative diagram of vector bundles on Y, with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_Y \arrow[r] \arrow[d, "\hat\Psi", "\cong"'] & T^*V|_Y \oplus \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_Y \arrow[r] \arrow[d, "\mathrm{d}\Psi|_Y^*", "\cong"'] & T^*V|_Y \arrow[r] \arrow[d, "\mathrm{id}", "\cong"'] & 0 \\
0 \arrow[r] & \nu^*|_Y \arrow[r] & T^*W|_Y \arrow[r] & T^*V|_Y \arrow[r] & 0.
\end{tikzcd} \tag{5.1}$$

Here $\mathrm{d}\Psi : TW|_{W'} \to \Psi^*\bigl(T(V \oplus \mathbb{C}^n)\bigr) = \Psi^*(TV) \oplus \Psi^*(T\mathbb{C}^n)$ is an isomorphism as Ψ is a local biholomorphism. Restricting to Y, taking duals, and noting that $\Psi|_Y : Y \to Y \times \{0\} \cong Y$ is the identity, gives an isomorphism $\mathrm{d}\Psi|_Y^* : T^*V|_Y \oplus \Psi^*(T^*\mathbb{C}^n)|_Y \to T^*W|_Y$. We write $\Psi^*(T^*\mathbb{C}^n)|_Y = \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_Y$, as it is a trivial vector bundle on Y with basis of sections $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$.

Thus, there is a unique isomorphism $\hat\Psi : \langle \mathrm{d}z_1, \ldots, \mathrm{d}z_n \rangle_Y \to \nu^*|_Y$ of vector bundles on Y which makes (5.1) *commute. This $\hat\Psi$ should identify $\mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n$ with $q|_Y$, for all such x, W', Ψ.*

We continue to use the notation introduced in Theorem 5.1 throughout the section; in particular let $\nu|_X$ and $q \in H^0(S^2\nu^*|_X)$ be as in Theorem 5.1(ii). Write $\Pi_{f,g} : F_{f,g} \to X$ for the *orthonormal frame bundle* of $(\nu|_X, q)$. That is, $\Pi_{f,g} : F_{f,g} \to X$ is a principal $\mathrm{O}(n, \mathbb{C})$-bundle over the complex analytic space X, and points of $\Pi_{f,g}^{-1}(x)$ in $F_{f,g}$ for $x \in X$ are bases $(e_1, \ldots, e_n)$ for $\nu|_x$ with $q|_x \cdot (e_i \otimes e_j) = \delta_{ij}$. The complex Lie group $\mathrm{O}(n, \mathbb{C})$ acts freely and transitively on $\Pi_{f,g}^{-1}(x)$ by $(A_{ij}) = (A_{ij})_{i,j=1}^n : (e_1, \ldots, e_n) \mapsto (\tilde e_1, \ldots, \tilde e_n)$ with $\tilde e_i = \sum_{j=1}^n A_{ij} e_j$ for $i = 1, \ldots, n$.

We have a normal subgroup $\mathrm{SO}(n, \mathbb{C}) \subset \mathrm{O}(n, \mathbb{C})$ of matrices (A_{ij}) with $\det(A_{ij}) = 1$, and $\mathrm{O}(n, \mathbb{C}) / \mathrm{SO}(n, \mathbb{C}) \cong \mathbb{Z}_2$. Write $P_{f,g} = F_{f,g} / \mathrm{SO}(n, \mathbb{C})$, and $\pi_{f,g} : P_{f,g} \to X$ for the natural projection. Then $\pi_{f,g} : P_{f,g} \to X$ is a principal $\mathbb{Z}_2$-bundle over X. Points of $\pi_{f,g}^{-1}(x)$ are *orientations* for the complex Euclidean vector space $(\nu_x, q|_x)$, and $P_{f,g}$ is the *orientation bundle* of (ν, q).

Theorem 5.2. **(a)** *Given V, W, f, g, X as in Theorem* 5.1, *and defining $P_{f,g}$ as above, there is a natural isomorphism of perverse sheaves on X:*

$$\Theta_{f,g} : \mathcal{PV}^\bullet_{V,f} \longrightarrow \mathcal{PV}^\bullet_{W,g} \otimes_{\mathbb{Z}_2} P_{f,g}, \tag{5.2}$$

where $\mathcal{PV}^\bullet_{V,f}, \mathcal{PV}^\bullet_{W,g}$ are the perverse sheaves of vanishing cycles from §2.5, *and if $\mathcal{F}^\bullet$ is a perverse sheaf on X then $\mathcal{F}^\bullet \otimes_{\mathbb{Z}_2} P_{f,g}$ means the perverse sheaf $\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{L}_{P_{f,g}}$, for $\mathcal{L}_{P_{f,g}} \in D^b_c(X)$ the rank one $\mathbb{Q}$-local system on X induced from $P_{f,g}$ by the nontrivial representation of $\mathbb{Z}_2$ on $\mathbb{Q}$. Also the following diagram commutes, where $\tau_{V,f}, \tau_{W,g}$ are the twisted monodromy operators from* (2.7):

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{V,f} \arrow[r, "\Theta_{f,g}"'] \arrow[d, "\tau_{V,f}"] & \mathcal{PV}^\bullet_{W,g} \otimes_{\mathbb{Z}_2} P_{f,g} \arrow[d, "\tau_{W,g}\otimes \mathrm{id}_{P_{f,g}}"'] \\
\mathcal{PV}^\bullet_{V,f} \arrow[r, "\Theta_{f,g}"] & \mathcal{PV}^\bullet_{W,g} \otimes_{\mathbb{Z}_2} P_{f,g}.
\end{tikzcd} \tag{5.3}$$

(b) *Now let $U \subseteq V$ be a complex submanifold, write $e := f|_U : U \to \mathbb{C}$, and suppose $\mathrm{Crit}(e) = X$ as complex analytic subspaces of V. Then as above we have principal $\mathbb{Z}_2$-bundles $P_{e,f}, P_{f,g}, P_{e,g}$ on X and isomorphisms of perverse sheaves $\Theta_{e,f}, \Theta_{f,g}, \Theta_{e,g}$ on X. There is a natural isomorphism of principal $\mathbb{Z}_2$-bundles $\Xi_{e,f,g} : P_{e,g} \to P_{f,g} \otimes_{\mathbb{Z}_2} P_{e,f}$ over X which makes the following commute:*

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{U,e} \arrow[r, "\Theta_{e,f}"'] \arrow[d, "\Theta_{e,g}"] & \mathcal{PV}^\bullet_{V,f} \otimes_{\mathbb{Z}_2} P_{e,f} \arrow[d, "\Theta_{f,g}\otimes \mathrm{id}_{P_{e,f}}"'] \\
\mathcal{PV}^\bullet_{W,g} \otimes_{\mathbb{Z}_2} P_{e,g} \arrow[r, "\mathrm{id}\otimes\Xi_{e,f,g}"] & \mathcal{PV}^\bullet_{W,g} \otimes_{\mathbb{Z}_2} P_{f,g} \otimes_{\mathbb{Z}_2} P_{e,f}.
\end{tikzcd} \tag{5.4}$$

Furthermore, if we have a submanifold $T \subseteq U$ and $d := e|_T : T \to \mathbb{C}$ with $\mathrm{Crit}(d) = X$, then the $\Xi_{e,f,g}$ have the associativity property

$$\begin{aligned} \big(\mathrm{id}_{P_{f,g}} \otimes \Xi_{d,e,f}\big) \circ \Xi_{d,f,g} = \big(\Xi_{e,f,g} \otimes \mathrm{id}_{P_{d,e}}\big) \circ \Xi_{d,e,g} :& \\ P_{d,g} \longrightarrow P_{f,g} \otimes_{\mathbb{Z}_2} P_{e,f} \otimes_{\mathbb{Z}_2} P_{d,e}.& \end{aligned} \tag{5.5}$$

(c) *The analogues of* **(a)**,**(b)** *hold for mixed Hodge modules: given V, W, f, g, X as in Theorem* 5.1, *there is a natural isomorphism in* $\mathrm{MHM}(X; T_s, N)$

$$\Theta^H_{f,g} : \mathcal{HV}^\bullet_{W,g} \longrightarrow \mathcal{HV}^\bullet_{V,f} \otimes_{\mathbb{Z}_2} P_{f,g}, \tag{5.6}$$

where if $M^\bullet \in D^b\,\mathrm{MHM}(X; T_s, N)$ then $M^\bullet \otimes_{\mathbb{Z}_2} P_{f,g}$ means $M^\bullet \overset{L}{\otimes} \mathcal{L}^H_{P_{f,g}}$, for $\mathcal{L}^H_{P_{f,g}}$ the unique object in $D^b\,\mathrm{MHM}(X; T_s, N)$ which is pure of weight 0, has trivial monodromy action, and has $\mathbf{rat}(\mathcal{L}^H_{P_{f,g}}) = \mathcal{L}_{P_{f,g}}$, for $\mathcal{L}_{P_{f,g}}$ as in **(a)**.

Also, these isomorphisms satisfy the compatibilities of **(b)**.

Theorems 5.1 and 5.2 will be proved in §5.1–§5.2 and §5.3–§5.5, respectively.

5.1 Theorem 5.1(i): g is locally $f \boxplus z_1^2 + \cdots + z_n^2$

We will prove Theorem 5.1(i) by induction on $n = \dim W - \dim V$. We start with the case $n = 1$. Suppose $\dim V = m$ and $\dim W = m + 1$, and choose local holomorphic coordinates $(y_1, \ldots, y_m, z)$ on W near x such that V is the hypersurface $z = 0$ near x, so $(y_1|_V, \ldots, y_m|_V)$ are local coordinates on V near x. Write $g = g(y_1, \ldots, y_m, z)$ and $f = f(y_1, \ldots, y_m)$, so that $f = g|_V$ means that $f(y_1, \ldots, y_m) = g(y_1, \ldots, y_m, 0)$.

Then $I_X = I_{(\mathrm{d}f)}$ near x is the ideal of holomorphic functions in $(y_1, \ldots, y_m)$ generated by $\frac{\partial f}{\partial y_j}$ for $j = 1, \ldots, m$, and $I_Y = I_{(\mathrm{d}g)}$ near x is the ideal of holomorphic functions in $(y_1, \ldots, y_m, z)$ generated by $\frac{\partial g}{\partial y_j}$ for $j = 1, \ldots, m$ and $\frac{\partial g}{\partial z}$. Since $X = Y$ as complex analytic subspaces, we have $I_X = I_Y|_{z=0}$, that is,

$$\begin{aligned}&\bigl(\tfrac{\partial f}{\partial y_j}(y_1, \ldots, y_m) : j = 1, \ldots, m\bigr) = \\ &\bigl(\tfrac{\partial g}{\partial y_j}(y_1, \ldots, y_m, 0) : j = 1, \ldots, m \quad \text{and} \quad \tfrac{\partial g}{\partial z}(y_1, \ldots, y_m, 0)\bigr).\end{aligned}$$

As $\frac{\partial g}{\partial y_j}(y_1, \ldots, y_m, 0) = \frac{\partial f}{\partial y_j}(y_1, \ldots, y_m)$, this holds provided $\frac{\partial g}{\partial z}(y_1, \ldots, y_m, 0)$ lies in $\bigl(\frac{\partial f}{\partial y_j}(y_1, \ldots, y_m) : j = 1, \ldots, m\bigr)$, that is, if and only if there exist holomorphic functions $a_j(y_1, \ldots, y_m)$ defined near x in V such that

$$\tfrac{\partial g}{\partial z}(y_1, \ldots, y_m, 0) = \textstyle\sum_{j=1}^m a_j(y_1, \ldots, y_m) \cdot \frac{\partial f}{\partial y_j}(y_1, \ldots, y_m).$$

Define new local holomorphic coordinates $(\tilde y_1, \ldots, \tilde y_m, \tilde z)$ on W near x by $\tilde y_j = y_j + a_j(y_1, \ldots, y_m)\, z$ and $\tilde z = z$. Then

$$\begin{aligned}\tfrac{\partial g}{\partial \tilde z}(\tilde y_1, \ldots, \tilde y_m, 0) &= \textstyle\sum_{j=1}^m \frac{\partial g}{\partial y_j}(y_1, \ldots, y_m, 0) \cdot \frac{\partial y_j}{\partial \tilde z} + \frac{\partial g}{\partial z}(y_1, \ldots, y_m, 0) \cdot \frac{\partial z}{\partial \tilde z} \\ &= \textstyle\sum_{j=1}^m \frac{\partial g}{\partial y_j}(y_1, \ldots, y_m, 0) \cdot (-a_j(y_1, \ldots, y_m)) \\ &\quad + \bigl(\textstyle\sum_{j=1}^m a_j(y_1, \ldots, y_m) \cdot \frac{\partial f}{\partial y_j}(y_1, \ldots, y_m)\bigr) \cdot 1 = 0.\end{aligned}$$

So we have $g(\tilde y_1, \ldots, \tilde y_m, 0) = f(\tilde y_1, \ldots, \tilde y_m)$ and $\frac{\partial g}{\partial \tilde z}(\tilde y_1, \ldots, \tilde y_m, 0) = 0$ in the new coordinates $(\tilde y_1, \ldots, \tilde y_m, \tilde z)$. Therefore we may write

$$g(\tilde y_1, \ldots, \tilde y_m, \tilde z) = f(\tilde y_1, \ldots, \tilde y_m) + \tilde z^2 h(\tilde y_1, \ldots, \tilde y_m, \tilde z), \tag{5.7}$$

for some holomorphic function h defined near x in W.

From (5.7), we see that $\mathrm{Crit}(f) = \mathrm{Crit}(g)$ as complex analytic subspaces of W is equivalent to $h \neq 0$ on $\mathrm{Crit}(f)$. So $h \neq 0$ near x. Hence we can choose an open neighbourhood W' of x in W, such that the coordinates $(\tilde y_1, \ldots, \tilde y_m, \tilde z)$ and function h are defined on W' and (5.7) holds there, and also $h \neq 0$ on W' and a holomorphic square root $h^{1/2}$ exists on W'. Defining functions $\hat y_1, \ldots, \hat y_m, \hat z$ on W' by $\hat y_j = \tilde y_j$ for $j = 1, \ldots, m$ and $\hat z = \tilde z\, h^{1/2}(\tilde y_1, \ldots, \tilde y_m, \tilde z)$, then $\hat y_1, \ldots, \hat y_m, \hat z$ are holomorphic coordinates on W'. All this holds provided W' is sufficiently small. In the new coordinates, by (5.7) we have

$$g(\hat y_1, \ldots, \hat y_m, \hat z) = f(\hat y_1, \ldots, \hat y_m) + \hat z^2.$$

Thus, if we define $\Psi : W' \to V \times \mathbb{C}$ by $\Psi(\hat{y}_1, \ldots, \hat{y}_m, \hat{z}) = \bigl((\hat{y}_1, \ldots, \hat{y}_m), \hat{z}\bigr)$, using $(\hat{y}_1|_V, \ldots, \hat{y}_m|_V)$ as local coordinates on V, then Ψ is a biholomorphism with an open neighbourhood of $(x, 0)$ in $V \times \mathbb{C}$, and $\Psi(v) = (v, 0)$ for $v \in W' \cap V$, and if $\Psi(w) = (v, z)$ for $w \in W'$ then $g(w) = f(v) + z^2$. This proves Theorem 5.1(i) when $n = \dim W - \dim V = 1$, and the first step in our induction.

For the inductive step, suppose $n \geqslant 1$ and Theorem 5.1(i) holds whenever $\dim W - \dim V \leqslant n$. Let X, V, W, f, g be as in Theorem 5.1 with $\dim W - \dim V = n + 1$, and $x \in X$. Since $X = \mathrm{Crit}(f) = \mathrm{Crit}(g)$ near x, as in (3.1) we have exact sequences

$$0 \longrightarrow T_xX \longrightarrow T_xV \xrightarrow[\mathrm{Hess}_x f]{} T_x^*V \longrightarrow T_x^*X \longrightarrow 0,$$

$$0 \longrightarrow T_xX \longrightarrow T_xW \xrightarrow{\mathrm{Hess}_x g} T_x^*W \longrightarrow T_x^*X \longrightarrow 0.$$

Since $\mathrm{Hess}_x f$ and $\mathrm{Hess}_x g$ have the same kernel, it follows that $\mathrm{Hess}_x g$ is the sum of $\mathrm{Hess}_x f$ and a nondegenerate quadratic form in the normal directions to T_xV in T_xW. Thus we can choose $u \in T_xW \setminus T_xV$ with $(u \otimes u) \cdot (\mathrm{Hess}_x g) \neq 0$.

Extend u to a holomorphic vector field $\tilde{u}$ defined near x on W such that $\tilde{u}|_x = u$. Define a local holomorphic function $h : W \to \mathbb{C}$ near x by $h = \tilde{u} \cdot \mathrm{d}g$. Then at x, using index notation for tensors, we have

$$u \cdot \mathrm{d}h|_x = u^a(\nabla_a \tilde{u}^b|_x)(\mathrm{d}_b g|_x) + u^a(\tilde{u}^b|_x)(\nabla_a \mathrm{d}_b g|_x) = 0 + (u \otimes u) \cdot (\mathrm{Hess}_x g) \neq 0.$$

Therefore $h = 0$ is a nonsingular hypersurface in W near x. Note too that X is a complex analytic subspace of this hypersurface $h = 0$ near x, as $h = \tilde{u} \cdot \mathrm{d}g$ and X is defined by $\mathrm{d}g = 0$.

Choose an open neighbourhood T of x in W such that h is defined on T and $h = 0$ is nonsingular there, and write $S = \{w \in T : h(w) = 0\}$. Then S is a complex submanifold of $T \subseteq W$ with $\dim S = \dim W - 1$, so that $\dim S - \dim V = n$. Write $e = g|_S : S \to \mathbb{C}$. As h is of the form $h = \tilde{u} \cdot \mathrm{d}g$, it follows that $\mathrm{Crit}(e) = \mathrm{Crit}(g) \cap T$ as complex analytic subspaces of T. Hence also $\mathrm{Crit}(e) = \mathrm{Crit}(f) \cap T$, as $\mathrm{Crit}(f) = \mathrm{Crit}(g)$ near x.

Applying Theorem 5.1(i) with $V \cap T, S, f|_{V \cap T}, e$ in place of V, W, f, g (allowed by induction as $\dim S - \dim(V \cap T) = n$) gives a local biholomorphism $\Psi_1 : S \to (V \cap T) \times \mathbb{C}^n$ near x mapping $\Psi_1 : s \mapsto \bigl(v, (z_1, \ldots, z_n)\bigr)$ with $e(s) = f(v) + z_1^2 + \cdots + z_n^2$. Similarly, applying Theorem 5.1(i) with $S, T, e, g|_T$ in place of V, W, f, g (allowed by the case $n = 1$ proved above) shows that near x there is a local biholomorphism $\Psi_2 : T \to S \times \mathbb{C}$ mapping $\Psi_2 : w \mapsto (s, z_{n+1})$ with $g(w) = e(s) + z_{n+1}^2$. Composing gives a local biholomorphism $\Psi = (\Psi_1 \times \mathrm{id}_{\mathbb{C}}) \circ \Psi_2 : T \to (V \cap T) \times \mathbb{C}^{n+1}$ mapping $\Psi : w \mapsto \bigl(v, (z_1, \ldots, z_n, z_{n+1})\bigr)$ with

$$g(w) = e(s) + z_{n+1}^2 = f(v) + z_1^2 + \cdots + z_n^2 + z_{n+1}^2.$$

This completes the inductive step, and proves Theorem 5.1(i).

5.2 Theorem 5.1(ii): the nondegenerate quadratic form q

If x, W', Ψ are as in Theorem 5.1(i), and $Y = W' \cap X$, then Theorem 5.1(ii) characterizes $q|_Y \in H^0(S^2\nu^*|_Y)$ as $q|_Y = \hat{\Psi}_*(\mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n)$. Note

too that this $q|_Y$ is a nondegenerate holomorphic quadratic form on $\nu|_Y$. Thus, for each $x \in X$ we have an open neighbourhood Y of x in X and a value for $q|_Y$. That is, we define $q|_Y$ on the sets Y in an open cover of X. As $S^2\nu^*$ is a sheaf, these restrictions determine q completely, provided they agree on overlaps $Y \cap Y'$ between different subsets Y, Y' in the open cover.

Suppose that $x, W', \Psi, Y, \hat\Psi, q|_Y = \hat\Psi_*(\mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n)$ and x', $W'', \Psi', Y', \hat\Psi', q|_{Y'} = \hat\Psi'_*(\mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n)$ are alternative choices above. Then, to prove Theorem 5.1(ii), it is enough to show that

$$\hat\Psi'_*(\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n)|_{Y\cap Y'} = \hat\Psi_*(\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n)|_{Y\cap Y'} \tag{5.8}$$

in $H^0(S^2\nu^*|_{Y\cap Y'})$. Combining (5.1) for W', Ψ, Y and W'', Ψ', Y' gives a commutative diagram of vector bundles on $Y \cap Y'$, with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{Y\cap Y'} \arrow[r] \arrow[d, "\cong"', "\hat\Psi^{-1}\circ\hat\Psi'"] & T^*V|_{Y\cap Y'} \oplus \langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{Y\cap Y'} \arrow[r] \arrow[d, "\cong"', "\mathrm{d}(\Psi'\circ\Psi^{-1})|^*_{Y\cap Y'}"] & T^*V|_{Y\cap Y'} \arrow[r] \arrow[d, "\cong"', "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{Y\cap Y'} \arrow[r] & T^*V|_{Y\cap Y'} \oplus \langle \mathrm{d}z_1,\ldots,\mathrm{d}z_n\rangle_{Y\cap Y'} \arrow[r] & T^*V|_{Y\cap Y'} \arrow[r] & 0. \end{tikzcd} \tag{5.9}$$

Here $\Psi' \circ \Psi^{-1}$ is a local biholomorphism $V \times \mathbb{C}^n \to V \times \mathbb{C}^n$ defined near $(Y \cap Y') \times \{0\}$, which is the identity on $V \times \{0\}$, and preserves the function $f \boxplus z_1^2 + \cdots + z_n^2 : V \times \mathbb{C}^n \to \mathbb{C}$. Therefore

$$\mathrm{d}(\Psi' \circ \Psi^{-1})|_{(Y\cap Y')\times\{0\}} : T(V \times \mathbb{C}^n)|_{(Y\cap Y')\times\{0\}} \longrightarrow T(V \times \mathbb{C}^n)|_{(Y\cap Y')\times\{0\}}$$

preserves $\mathrm{Hess}\bigl(f \boxplus z_1^2+\cdots+z_n^2\bigr)$ in $H^0\bigl(S^2T^*(V\times\mathbb{C}^n)|_{(Y\cap Y')\times\{0\}}\bigr)$. As $\mathrm{Hess}\bigl(f \boxplus z_1^2 + \cdots + z_n^2\bigr) \cong \mathrm{Hess}\, f + \mathrm{d}z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n$, from (5.9) we see that $\hat\Psi^{-1} \circ \hat\Psi'$ preserves $z_1 \otimes \mathrm{d}z_1 + \cdots + \mathrm{d}z_n \otimes \mathrm{d}z_n$. That is, we have

$$(\hat\Psi^{-1}\circ\hat\Psi')_*(\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n)|_{Y\cap Y'} = (\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n)|_{Y\cap Y'}.$$

Composing with $\hat\Psi_*$ gives (5.8), and proves Theorem 5.1(ii).

5.3 Theorem 5.2(a): the isomorphism $\Theta_{f,g}$

We now prove Theorem 5.2(a), using the notation introduced above. For each $x \in X$, choose $\Psi_x : U_x \xrightarrow{\cong} \tilde U_x$ with $x \in U_x$ as in Theorem 5.1(i), and define $X'_x, \hat\Psi_x$ as in Theorem 5.1(ii), where we now use 'x' as a subscript to distinguish choices for different $x \in X$.

Identifying $X'_x \subseteq V \subseteq W$ and $X'_x \times \{0\} \subseteq V \times \mathbb{C}^n$ in the obvious way, define an isomorphism $\Theta_x : \mathcal{PV}^\bullet_{V,f}|_{X'_x} \to \mathcal{PV}^\bullet_{W,g} \otimes_{\mathbb{Z}_2} P_{f,g}|_{X'_x}$ in $\mathrm{Perv}(X'_x)$ by the commutative diagram:

$$\begin{tikzcd} \mathcal{PV}^\bullet_{V,f}|_{X'_x} \arrow[rr, "\alpha_x"'] \arrow[dd, "\Theta_x"] & & \mathcal{PV}^\bullet_{V,f}|_{X'_x} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n, z_1^2+\cdots+z_n^2} \arrow[d, "\beta_x"'] \\ & & \mathcal{PV}^\bullet_{V\times\mathbb{C}^n, f\boxplus z_1^2+\cdots+z_n^2}\bigr|_{X'_x\times\{0\}} \arrow[d, "\gamma_x"'] \\ \mathcal{PV}^\bullet_{W,g} \otimes_{\mathbb{Z}_2} P_{f,g}\bigr|_{X'_x} & & \mathcal{PV}^\bullet_{W,g}|_{X'_x}. \arrow[ll, "\delta_x"'] \end{tikzcd} \tag{5.10}$$

Here the isomorphisms $\alpha_x,\ldots,\delta_x$ in (5.10) are defined as follows:

(i) α_x comes from the isomorphism $\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}\cong\mathbb{Q}_{\{0\}}$ in (2.8).

(ii) β_x comes from the Thom–Sebastiani Theorem for $\mathcal{PV}^\bullet_{V,f}$, Theorem 2.15.

(iii) γ_x is defined in a similar way to Φ_* in (2.10) for $\Psi_x:W'_x\to V\times\mathbb{C}^n$.

(iv) The isomorphism $\hat\Psi_x$ in (5.1) induces a trivialization of $(\nu|_{X'_x},q|_{X'_x})$, and hence of $P_{f,g}|_{X'_x}\to X'_x$, and δ_x is induced by this trivialization.

Now let $x,y\in X$. Since α_x,β_x in (i),(ii) are independent of x,W'_x,Ψ_x except in their choice of domain X'_x, we have

$$\alpha_x|_{X'_x\cap X'_y}=\alpha_y|_{X'_x\cap X'_y}\quad\text{and}\quad\beta_x|_{X'_x\cap X'_y}=\beta_y|_{X'_x\cap X'_y}.\tag{5.11}$$

Consider $\Psi_y^{-1}\circ\Psi_x|_{\Psi_x^{-1}(\operatorname{Im}\Psi_y)}:\Psi_x^{-1}(\operatorname{Im}\Psi_y)\to\Psi_y^{-1}(\operatorname{Im}\Psi_x)$. This is a local biholomorphism of open subsets of W defined near $X'_x\cap X'_y$, which is the identity on $V\cap\Psi_x^{-1}(\operatorname{Im}\Psi_y)$ and hence on $X'_x\cap X'_y$. Also

$$g\circ(\Psi_y^{-1}\circ\Psi_x)|_{\Psi_x^{-1}(\operatorname{Im}\Psi_y)}=(f\boxplus z_1^2+\cdots+z_n^2)\circ\Psi_x|_{\Psi_x^{-1}(\operatorname{Im}\Psi_y)}=g|_{\Psi_x^{-1}(\operatorname{Im}\Psi_y)}.$$

Therefore Theorem 3.1 shows $\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}:TW|_{X'_x\cap X'_y}\to TW|_{X'_x\cap X'_y}$ has $\det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr)=\pm1$, and

$$\begin{aligned}\bigl(\Psi_y^{-1}\circ\Psi_x|_{\Psi_x^{-1}(\operatorname{Im}\Psi_y)}\bigr)_*=\det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr)\cdot:\\ \mathcal{PV}^\bullet_{W,g}|_{X'_x\cap X'_y}\longrightarrow\mathcal{PV}^\bullet_{W,g}|_{X'_x\cap X'_y}.\end{aligned}$$

Thus $\gamma_x=(\Psi_x)_*$, $\gamma_y=(\Psi_y)_*$ and functoriality of pushforwards implies that

$$\begin{aligned}\gamma_x|_{X'_x\cap X'_y}&=\gamma_y|_{X'_x\cap X'_y}\circ\bigl(\Psi_y^{-1}\circ\Psi_x|_{\Psi_x^{-1}(\operatorname{Im}\Psi_y)}\bigr)_*\\&=\det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr)\cdot\gamma_y|_{X'_x\cap X'_y}.\end{aligned}\tag{5.12}$$

The morphisms $\delta_x|_{X'_x\cap X'_y},\delta_y|_{X'_x\cap X'_y}$ come from trivializations of $P_{f,g}|_{X'_x\cap X'_y}$ induced by trivializations of $(\nu|_{X'_x\cap X'_y},q|_{X'_x\cap X'_y})$ which differ by the orthogonal transformation

$$\bigl(\hat\Psi_x\circ\hat\Psi_y^{-1}\bigr)^*:\nu|_{X'_x\cap X'_y}\longrightarrow\nu|_{X'_x\cap X'_y}.$$

It follows that

$$\delta_y|_{X'_x\cap X'_y}=\det\bigl(\hat\Psi_x\circ\hat\Psi_y^{-1}\bigr)^*\cdot\delta_x|_{X'_x\cap X'_y}.\tag{5.13}$$

We have an exact sequence

$$0\longrightarrow TV|_{X'_x\cap X'_y}\longrightarrow TW|_{X'_x\cap X'_y}\longrightarrow\nu|_{X'_x\cap X'_y}\longrightarrow0.$$

Choosing a compatible splitting $TW|_{X'_x\cap X'_y}\cong TV|_{X'_x\cap X'_y}\oplus\nu|_{X'_x\cap X'_y}$, we identify

$$\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\cong\begin{pmatrix}\mathrm{id}_{TV|_{X'_x\cap X'_y}} & *\\ 0 & (\hat\Psi_x\circ\hat\Psi_y^{-1})^*\end{pmatrix}.$$

Therefore

$$\det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr)=\det\bigl(\hat\Psi_x\circ\hat\Psi_y^{-1}\bigr)^*. \tag{5.14}$$

Combining (5.10)–(5.14), we see that

$$\Theta_x|_{X'_x\cap X'_y}=(\delta_x\circ\gamma_x\circ\beta_x\circ\alpha_x)|_{X'_x\cap X'_y}=(\delta_y\circ\gamma_y\circ\beta_y\circ\alpha_y)|_{X'_x\cap X'_y}=\Theta_y|_{X'_x\cap X'_y}.$$

We have chosen an open cover $\{X'_x : x\in X\}$ for X, and on each X'_x we have defined an isomorphism $\Theta_x:\mathcal{PV}^\bullet_{V,f}|_{X'_x}\to\mathcal{PV}^\bullet_{W,g}\otimes_{\mathbb{Z}_2}P_{f,g}|_{X'_x}$ of perverse sheaves, such that on overlaps $X'_x\cap X'_y$ we have $\Theta_x|_{X'_x\cap X'_y}=\Theta_y|_{X'_x\cap X'_y}$. Therefore by Theorem 2.5(i), there exists a unique isomorphism $\Theta_{f,g}$ as in (5.2) such that $\Theta_{f,g}|_{X'_x}=\Theta_x$ for all $x\in X$.

This morphism $\Theta_{f,g}$ is independent of the choices of Ψ_x, W'_x for each $x\in X$, as given a second set of choices $\dot\Psi_x,\dot W'_x$ and defining $\dot X'_x,\dot\Theta_x,\dot\Theta_{f,g}$ in the obvious way, the proof above also shows that

$$\Theta_{f,g}|_{X'_x\cap\dot X'_x}=\Theta_x|_{X'_x\cap\dot X'_x}=\dot\Theta_x|_{X'_x\cap\dot X'_x}=\dot\Theta_{f,g}|_{X'_x\cap\dot X'_x},$$

so $\Theta_{f,g}=\dot\Theta_{f,g}$ as $\{X'_x\cap\dot X'_x : x\in X\}$ is an open cover of X.

To see that (5.3) commutes, consider how twisted monodromy operators act in (5.10). Since $\tau_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}=\mathrm{id}$ by (2.9), we see that

$$\alpha_x\circ\tau_{V,f}|_{X'_x}=\bigl(\tau_{V,f}|_{X'_x}\overset{L}{\boxtimes}\tau_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}\bigr)\circ\alpha_x. \tag{5.15}$$

Theorem 2.15 implies that

$$\beta_x\circ\bigl(\tau_{V,f}|_{X'_x}\overset{L}{\boxtimes}\tau_{\mathbb{C}^n,z_1^2+\cdots+z_n^2}\bigr)=\tau_{V\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}|_{X'_x\times\{0\}}\circ\beta_x. \tag{5.16}$$

As pushforwards in Definition 2.17 commute with monodromy, we have

$$\gamma_x\circ\tau_{V\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}|_{X'_x\times\{0\}}=\tau_{W,g}|_{X'_x}\circ\gamma_x. \tag{5.17}$$

And since monodromy does not affect the trivialization of $P_{f,g}|_{X'_x}$ used to define δ_x, we have

$$\delta_x\circ\tau_{W,g}|_{X'_x}=\bigl(\tau_{W,g}\otimes\mathrm{id}_{P_{f,g}}\bigr)|_{X'_x}\circ\delta_x. \tag{5.18}$$

Combining (5.10) and (5.15)–(5.18) shows that the restriction of (5.3) to X'_x commutes. As the X'_x for $x\in X$ cover X, we see using Theorem 2.5(i) that (5.3) commutes. This proves Theorem 5.2(a).

5.4 Theorem 5.2(b): composition of the $\Theta_{f,g}$

Let U,V,W,e,f,g,X be as in Theorem 5.2(b).

To distinguish the various normal bundles, write $\nu_{UV},\nu_{UW},\nu_{VW}$ for the normal bundles of U in V, U in W and V in W, respectively. Then $X\subseteq U\subseteq V\subseteq W$, and we have a natural exact sequence of vector bundles on X:

$$0\longrightarrow\nu_{UV}|_X\longrightarrow\nu_{UW}|_X\longrightarrow\nu_{VW}|_X\longrightarrow 0. \tag{5.19}$$

Applying Theorem 5.1(ii) for U,V,e,f and U,W,e,g and V,W,f,g gives nondegenerate quadratic forms on $\nu_{UV}|_X,\nu_{UW}|_X,\nu_{VW}|_X$, which to distinguish them we write as $q_{ef}\in H^0\bigl(S^2\nu_{UV}^*|_X\bigr)$, $q_{eg}\in H^0\bigl(S^2\nu_{UW}^*|_X\bigr)$, $q_{fg}\in H^0\bigl(S^2\nu_{VW}^*|_X\bigr)$.

Let $m=\dim V-\dim U$ and $n=\dim W-\dim V$. By Theorem 5.1(ii), there exist local biholomorphisms $\Psi_{UV}:V\rightarrow U\times\mathbb{C}^m$ identifying $f\cong e\boxplus y_1^2+\cdots+y_m^2$, where $(y_1,\ldots,y_m)$ are coordinates on $\mathbb{C}^m$, and Ψ_{UV} identifies q_{ef} with $\mathrm{d}y_1\otimes\mathrm{d}y_1+\cdots+\mathrm{d}y_m\otimes\mathrm{d}y_m$. Similarly, there exist local biholomorphisms $\Psi_{VW}:W\rightarrow V\times\mathbb{C}^n$ identifying $g\cong f\boxplus z_1^2+\cdots+z_n^2$, where $(z_1,\ldots,z_n)$ are coordinates on $\mathbb{C}^n$, and Ψ_{VW} identifies q_{fg} with $\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n$.

Composing gives a local biholomorphism $(\Psi_{UV}\times\mathrm{id}_{\mathbb{C}^n})\circ\Psi_{VW}:W\rightarrow U\times\mathbb{C}^{m+n}$ identifying $g\cong e\boxplus y_1^2+\cdots+y_m^2+z_1^2+\cdots+z_n^2$, where $(y_1,\ldots,y_m,z_1,\ldots,z_n)$ are the coordinates on $\mathbb{C}^{m+n}$. But the characterization of q_{eg} in Theorem 5.1(ii) therefore shows that $(\Psi_{UV}\times\mathrm{id}_{\mathbb{C}^n})\circ\Psi_{VW}$ identifies q_{eg} with $\mathrm{d}y_1\otimes\mathrm{d}y_1+\cdots+\mathrm{d}y_m\otimes\mathrm{d}y_m+\mathrm{d}z_1\otimes\mathrm{d}z_1+\cdots+\mathrm{d}z_n\otimes\mathrm{d}z_n$. From this we see that locally near each $x\in X$, there exists a splitting

$$\nu_{UW}|_X\cong\nu_{UV}|_X\oplus\nu_{VW}|_X \tag{5.20}$$

compatible with (5.19), which identifies

$$\begin{split}&q_{eg}\in H^0\bigl(S^2\nu_{UW}^*|_X\bigr)\cong\\ &q_{ef}\oplus q_{fg}\oplus 0\in H^0\bigl(S^2\nu_{UV}^*|_X\bigr)\oplus H^0\bigl(S^2\nu_{VW}^*|_X\bigr)\oplus H^0\bigl(\nu_{UV}^*\otimes\nu_{VW}^*|_X\bigr).\end{split} \tag{5.21}$$

In fact, as q_{eg} is nondegenerate, the local splittings (5.20) satisfying (5.21) are unique, since we can define the complementary bundle $\nu_{VW}|_X\subset\nu_{UW}|_X$ to be the orthogonal bundle of $\nu_{UV}|_X$ in $\nu_{UW}|_X$ with respect to the complex Euclidean metric q_{eg}. So the local splittings glue to give a unique global splitting (5.20) on X for which (5.21) holds. Equations (5.20)–(5.21) now imply that the frame bundles $F_{e,f},F_{e,g},F_{f,g}$ are related by

$$F_{e,g}\cong\bigl(\mathrm{O}(m+n,\mathbb{C})\times F_{e,f}\times_X F_{f,g}\bigr)/\bigl(\mathrm{O}(m,\mathbb{C})\times\mathrm{O}(n,\mathbb{C})\bigr),$$

where $\mathrm{O}(m,\mathbb{C})\times\mathrm{O}(n,\mathbb{C})$ is included as the subgroup of $\mathrm{O}(m+n,\mathbb{C})$ preserving the splitting $\mathbb{C}^{m+n}=\mathbb{C}^m\oplus\mathbb{C}^n$ in the obvious way, and $\mathrm{O}(m,\mathbb{C}),\mathrm{O}(n,\mathbb{C})$ act on $F_{e,f},F_{f,g}$ by the principal bundle actions. Thus we have natural isomorphisms

$$\begin{aligned}P_{e,g}&=F_{e,g}/\mathrm{SO}(m+n,\mathbb{C})\\ &\cong\bigl[\bigl(\mathrm{O}(m+n,\mathbb{C})\times F_{e,f}\times_X F_{f,g}\bigr)/\bigl(\mathrm{O}(m,\mathbb{C})\times\mathrm{O}(n,\mathbb{C})\bigr)\bigr]/\mathrm{SO}(m+n,\mathbb{C})\\ &\cong\bigl((\mathrm{O}(m+n,\mathbb{C})/\mathrm{SO}(m+n,\mathbb{C}))\times F_{e,f}\times_X F_{f,g}\bigr)/\bigl(\mathrm{O}(m,\mathbb{C})\times\mathrm{O}(n,\mathbb{C})\bigr)\\ &\cong\bigl((\mathbb{Z}_2\times F_{e,f}\times_X F_{f,g}\bigr)/\bigl((\mathbb{Z}_2\ltimes\mathrm{SO}(m,\mathbb{C}))\times(\mathbb{Z}_2\ltimes\mathrm{SO}(n,\mathbb{C}))\bigr)\\ &\cong\bigl(\mathbb{Z}_2\times(F_{e,f}/\mathrm{SO}(m,\mathbb{C}))\times_X(F_{f,g}/\mathrm{SO}(n,\mathbb{C}))\bigr)/\bigl(\mathbb{Z}_2\times\mathbb{Z}_2\bigr)\\ &=\bigl(\mathbb{Z}_2\times P_{e,f}\times_X P_{f,g}\bigr)/\bigl(\mathbb{Z}_2\times\mathbb{Z}_2\bigr)\\ &\cong\bigl(P_{e,f}\times_X P_{f,g}\bigr)/\mathbb{Z}_2\cong P_{f,g}\otimes_{\mathbb{Z}_2}P_{e,f}.\end{aligned} \tag{5.22}$$

Composing gives the isomorphism $\Xi_{e,f,g}:P_{e,g}\rightarrow P_{f,g}\otimes_{\mathbb{Z}_2}P_{e,f}$ that we want.

To prove (5.4) commutes, using the notation of §5.3, let $\Theta_{f,g}$ be defined using data $\Psi_x : W'_x \to V \times \mathbb{C}^n$, $X'_x, \hat{\Psi}_x, \Theta_x, \alpha_x, \ldots, \delta_x$ for $x \in X$ with $x \in W'_x \subseteq W$, and let $\Theta_{e,f}$ be defined using data $\dot{\Psi}_x : \dot{V}'_x \to U \times \mathbb{C}^m$, $\dot{X}'_x, \dot{\hat{\Psi}}_x, \dot{\Theta}_x, \dot{\alpha}_x, \ldots, \dot{\delta}_x$ for $x \in X$ with $x \in \dot{V}'_x \subseteq V$. For each $x \in X$, define open $x \in \ddot{W}'_x \subseteq W$ by $\ddot{W}'_x = \Psi_x^{-1}(\dot{V}'_x \times \mathbb{C}^n)$, and define $\ddot{\Psi}_x : \ddot{W}'_x \to U \times \mathbb{C}^{m+n}$ by $\ddot{\Psi}_x = (\dot{\Psi}_x \times \mathrm{id}_{\mathbb{C}^n}) \circ \Psi_x|_{\ddot{W}'_x}$. That is, if $w \in \ddot{W}'_x$ and $\Psi_x(w) = \bigl(v, (z_1, \ldots, z_n)\bigr)$ with $\dot{\Psi}_x(v) = \bigl(u, (y_1, \ldots, y_m)\bigr)$, then $\ddot{\Psi}_x(v) = \bigl(u, (y_1, \ldots, y_m, z_1, \ldots, z_n)\bigr)$.

Then $\ddot{\Psi}_x : \ddot{W}'_x \to U \times \mathbb{C}^{m+n}$ satisfies the conditions of Theorem 5.1(i) for x, U, W, e, g. Thus, $\ddot{\Psi}_x, \ddot{W}'_x$ for all $x \in X$ are possible choices in the definition of $\Theta_{e,g}$ in §5.3. As $\Theta_{e,g}$ is independent of these choices, we take $\Theta_{e,g}$ to be defined using $\ddot{\Psi}_x, \ddot{W}'_x$, and use the obvious notation $\ddot{X}'_x, \ddot{\hat{\Psi}}_x, \ddot{\Theta}_x, \ddot{\alpha}_x, \ldots, \ddot{\delta}_x$ for $x \in X$. Then $\ddot{X}'_x = X'_x \cap \dot{X}'_x$.

For each $x \in X$, we now have a commutative diagram in $\mathrm{Perv}(\ddot{X}'_x)$:

$$
\begin{tikzcd}
\mathcal{PV}^\bullet_{W,g}|_{\ddot{X}'_x} & & & \mathcal{PV}^\bullet_{U\times\mathbb{C}^{m+n},e\boxplus y_1^2+\cdots+y_m^2+z_1^2+\cdots+z_n^2}|_{\ddot{X}'_x\times\{0\}} \arrow[lll, "\ddot{\gamma}_x"'] \\
\mathcal{PV}^\bullet_{V\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}|_{\ddot{X}'_x\times\{0\}} \arrow[u, "\gamma_x|_{\ddot{X}'_x}"] & & & \mathcal{PV}^\bullet_{U,e}|_{\ddot{X}'_x} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^{m+n},y_1^2+\cdots+y_m^2+z_1^2+\cdots+z_n^2} \arrow[u, "\ddot{\beta}_x"'] \\
& \mathcal{PV}^\bullet_{U\times\mathbb{C}^m,e\boxplus y_1^2+\cdots+y_m^2}|_{\ddot{X}'_x\times\{0\}} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \arrow[uurr] \arrow[dl, "\dot{\gamma}_x|_{\ddot{X}'_x} \overset{L}{\boxtimes} \mathrm{id}"] & \mathcal{PV}^\bullet_{U,e}|_{\ddot{X}'_x} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^m,y_1^2+\cdots+y_m^2} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \arrow[l, "\dot{\beta}_x|_{\ddot{X}'_x} \overset{L}{\boxtimes} \mathrm{id}"] \arrow[ur] & \\
\mathcal{PV}^\bullet_{V,f}|_{\ddot{X}'_x} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \arrow[uu, "\beta_x|_{\ddot{X}'_x}"] & & & \mathcal{PV}^\bullet_{U,e}|_{\ddot{X}'_x} \arrow[uu, "\ddot{\alpha}_x"'] \arrow[ul] \arrow[d, "\dot{\alpha}_x|_{\ddot{X}'_x}"'] \\
\mathcal{PV}^\bullet_{V,f}|_{\ddot{X}'_x} \arrow[u, "\alpha_x|_{\ddot{X}'_x}"] & \mathcal{PV}^\bullet_{U\times\mathbb{C}^m,e\boxplus y_1^2+\cdots+y_m^2}|_{\ddot{X}'_x\times\{0\}} \arrow[l, "\dot{\gamma}_x|_{\ddot{X}'_x}"'] \arrow[uu] & & \mathcal{PV}^\bullet_{U,e}|_{\ddot{X}'_x} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^m,y_1^2+\cdots+y_m^2}, \arrow[ll, "\dot{\beta}_x|_{\ddot{X}'_x}"']
\end{tikzcd}
$$

where the top right quadrilateral commutes because of associativity in the Thom–Sebastiani Theorem for $\mathcal{PV}^\bullet_{V,f}$, Theorem 2.15. Therefore

$$
\ddot{\gamma}_x \circ \ddot{\beta}_x \circ \ddot{\alpha}_x = \bigl(\gamma_x \circ \beta_x \circ \alpha_x\bigr)\big|_{\ddot{X}'_x} \circ \bigl(\dot{\gamma}_x \circ \dot{\beta}_x \circ \dot{\alpha}_x\bigr)\big|_{\ddot{X}'_x}, \tag{5.23}
$$

where $\alpha_x, \dot{\alpha}_x, \ddot{\alpha}_x, \ldots$ are as in (5.10) in the definitions of $\Theta_{f,g}, \Theta_{e,f}, \Theta_{e,g}$.

For the trivializations of principal $\mathbb{Z}_2$-bundles used to define δ_x in §5.3, we

have a diagram in $\mathrm{Perv}(\ddot X'_x)$:

$$\begin{tikzcd}
\mathcal{PV}^\bullet_{W,g}\big\vert_{\ddot X'_x} \arrow[rr, "\ddot\delta_x"'] \arrow[d, "\delta_x\vert_{\ddot X'_x}"] && \mathcal{PV}^\bullet_{W,g}\otimes_{\mathbb{Z}_2} P_{e,g}\big\vert_{\ddot X'_x} \arrow[d, "\mathrm{id}\otimes\Xi_{e,f,g}\vert_{\ddot X'_x}"'] \\
\mathcal{PV}^\bullet_{W,f}\otimes_{\mathbb{Z}_2} P_{f,g}\big\vert_{\ddot X'_x} \arrow[d, "\Theta_{f,g}^{-1}\vert_{\ddot X'_x}"] && \mathcal{PV}^\bullet_{W,g}\otimes_{\mathbb{Z}_2} P_{f,g}\otimes_{\mathbb{Z}_2} P_{e,f}\big\vert_{\ddot X'_x} \\
\mathcal{PV}^\bullet_{V,f}\vert_{\ddot X'_x} \arrow[rr, "\dot\delta_x\vert_{\ddot X'_x}"] && \mathcal{PV}^\bullet_{V,f}\otimes_{\mathbb{Z}_2} P_{e,f}\big\vert_{\ddot X'_x}, \arrow[u, "\Theta_{f,g}\otimes\mathrm{id}_{P_{e,f}}\vert_{\ddot X'_x}"']
\end{tikzcd} \tag{5.24}$$

which commutes because the trivializations of $P_{f,g}|_{X'_x}, P_{e,f}|_{\dot X'_x}, P_{e,g}|_{\ddot X'_x}$ used to define $\delta_x, \dot\delta_x, \ddot\delta_x$ are compatible with $\Xi_{e,f,g}|_{\ddot X'_x}$. Thus we have

$$\begin{aligned}
\big[(\mathrm{id}\otimes\Xi_{e,f,g})\circ\Theta_{e,g}\big]\big\vert_{\ddot X'_x} &= \big[(\mathrm{id}\otimes\Xi_{e,f,g})|_{\ddot X'_x}\circ\ddot\delta_x\big]\circ\big[\ddot\gamma_x\circ\ddot\beta_x\circ\ddot\alpha_x\big]\\
&= \big[\big(\Theta_{f,g}\otimes\mathrm{id}_{P_{e,f}}\big)|_{\ddot X'_x}\circ\dot\delta_x|_{\ddot X'_x}\circ\Theta_{f,g}^{-1}|_{\ddot X'_x}\circ\delta_x|_{\ddot X'_x}\big]\\
&\qquad\circ\big[\big(\gamma_x\circ\beta_x\circ\alpha_x\big)\big\vert_{\ddot X'_x}\circ\big(\dot\gamma_x\circ\dot\beta_x\circ\dot\alpha_x\big)\big\vert_{\ddot X'_x}\big]\\
&= \big(\Theta_{f,g}\otimes\mathrm{id}_{P_{e,f}}\big)|_{\ddot X'_x}\circ\dot\delta_x|_{\ddot X'_x}\circ\Theta_{f,g}^{-1}|_{\ddot X'_x}\circ\Theta_{f,g}|_{\ddot X'_x}\circ\big(\dot\gamma_x\circ\dot\beta_x\circ\dot\alpha_x\big)\big\vert_{\ddot X'_x}\\
&= \big[\big(\Theta_{f,g}\otimes\mathrm{id}_{P_{e,f}}\big)|_{\ddot X'_x}\circ\big(\dot\delta_x\circ\dot\gamma_x\circ\dot\beta_x\circ\dot\alpha_x\big)\big\vert_{\ddot X'_x}\big]\\
&= \big[\big(\Theta_{f,g}\otimes\mathrm{id}_{P_{e,f}}\big)\circ\Theta_{e,f}\big]\big\vert_{\ddot X'_x}.
\end{aligned}\tag{5.25}$$

Here in the first, third and fifth steps we use (5.10) for $\Theta_{e,g}, \Theta_{f,g}, \Theta_{e,f}$ and $\Theta_x = \Theta_{f,g}|_{X_x}$, and so on, and in the second we substitute in (5.23) and (5.24). Equation (5.25) implies that (5.4) commutes when restricted to $\ddot X'_x$. As $\{\ddot X'_x : x \in X\}$ is an open cover of X, equation (5.4) commutes by Theorem 2.5(i).

Finally, equation (5.5) follows from naturalness of the isomorphisms used to define $\Xi_{e,f,g}$ in (5.22). It is basically to do with associativity in the splitting $\mathbb{C}^{l+m+n} = (\mathbb{C}^l \oplus \mathbb{C}^m) \oplus \mathbb{C}^n = \mathbb{C}^l \oplus (\mathbb{C}^m \oplus \mathbb{C}^n)$, where $l = \dim U - \dim T$, $m = \dim V - \dim U$ and $n = \dim W - \dim V$. This proves Theorem 5.2(b).

5.5 Theorem 5.2(c): the case of mixed Hodge modules

Given all our previous assumptions and notation, the definition of the isomorphism $\Theta^H_{f,g}$ in (5.7) is a straightforward extension of (5.10), using the isomorphism $\mathcal{HV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \cong \mathbb{Q}^H_{\{0\}}$ of (2.17), Theorem 2.23, and the geometric facts already established. By the commutative diagram (5.3), the isomorphism $\Theta^H_{f,g}$ can be lifted to an isomorphism in the category $\mathrm{MHM}(X;T_s,N)$, taking into account the monodromy action also. This proves the mixed Hodge module analogue of Theorem 5.2(a). The analogue of (b) then follows from the results for perverse sheaves already proved, the sheaf property of mixed Hodge modules with monodromy, and faithfulness of the realization functor **rat** on mixed Hodge modules. The proof of Theorem 5.2 is complete.

6 Comparing $\mathcal{PV}^\bullet_{V,f}, \mathcal{PV}^\bullet_{W,g}$ when $\mathrm{Crit}(f) \cong \mathrm{Crit}(g)$

We will divide our fourth main result, which answers Question 1.1(d), into Theorems 6.1, 6.4 and 6.7, which are proved in §6.1, §6.2–§6.4 and §6.5–§6.7 respectively. The main themes are these. Suppose X is a complex analytic space, V, W are complex manifolds, $f : V \to \mathbb{C}$ and $g : W \to \mathbb{C}$ are holomorphic functions, and $j : X \to \mathrm{Crit}(f)$, $k : X \to \mathrm{Crit}(g)$ are isomorphisms of complex analytic spaces. We want to relate the perverse sheaves of vanishing cycles $j^*(\mathcal{PV}^\bullet_{V,f})$ and $k^*(\mathcal{PV}^\bullet_{W,g})$ on X.

Theorem 6.1 constructs a principal $\mathbb{Z}_2$-bundle $Q_{f,g} \to X$ which is similar to $P_{f,g} \to X$ in Theorems 5.1–5.2. Theorem 6.4 defines a notion of *compatibility* for the triples (V, f, j) and (W, g, k), which is an equivalence relation. It shows that if $(V, f, j), (W, g, k)$ are compatible and $\dim W - \dim V = n \geqslant 0$, then there exist local biholomorphisms $\Psi : W \to V \times \mathbb{C}^n$ which identify g with $f \boxplus z_1^2 + \cdots + z_n^2$ and k with $j \times 0$. Theorem 6.7 shows that for compatible $(V, f, j), (W, g, k)$ there is a canonical, functorial isomorphism $\Delta_{f,g} : j^*(\mathcal{PV}^\bullet_{V,f}) \to k^*(\mathcal{PV}^\bullet_{W,g}) \otimes_{\mathbb{Z}_2} Q_{f,g}$.

Theorem 6.1. *Let X be a fixed complex analytic space. In all of this section, a **triple** (V, f, j) means a complex manifold V, a holomorphic function $f : V \to \mathbb{C}$ so that $\mathrm{Crit}(f) \subseteq V$ is a closed complex analytic subspace, and an isomorphism of complex analytic spaces $j : X \to \mathrm{Crit}(f)$. Write $K_V = \Lambda^{\dim V} T^*V$ for the canonical bundle of V and $K_V^{\otimes^2} = K_V \otimes K_V$. Then $j^*(K_V)$ and $j^*\big(K_V^{\otimes^2}\big)$ are holomorphic line bundles on X.*

(i) *Let (V, f, j) and (W, g, k) be triples. As in §3.1 we have exact sequences of coherent sheaves on X:*

$$
\begin{aligned}
&0 \longrightarrow TX \longrightarrow j^*(TV) \xrightarrow[j^*(\mathrm{Hess}\, f)]{} j^*(T^*V) \longrightarrow T^*X \longrightarrow 0,\\
&0 \longrightarrow TX \longrightarrow k^*(TW) \xrightarrow{k^*(\mathrm{Hess}\, g)} k^*(T^*W) \longrightarrow T^*X \longrightarrow 0.
\end{aligned}
\tag{6.1}
$$

Properties of determinant line bundles therefore give canonical isomorphisms

$$
\begin{aligned}
\alpha &: \det(TX) \otimes j^*(K_V) \longrightarrow j^*(K_V)^* \otimes \det(T^*X),\\
\beta &: \det(TX) \otimes k^*(K_W) \longrightarrow k^*(K_W)^* \otimes \det(T^*X),
\end{aligned}
\tag{6.2}
$$

where $\det(TX), \det(T^*X)$ are the determinant line bundles of the analytic coherent sheaves TX, T^*X, noting that $\det(j^*(T^*V)) = \Lambda^{\dim V}\big(j^*(T^*V)\big) \cong j^*(K_V)$ and $\det(j^*(TV)) = \Lambda^{\dim V}\big(j^*(TV)\big) \cong j^*(K_V)^*$, and similarly for W.

Define an isomorphism $\Upsilon_{f,g} : j^*\big(K_V^{\otimes^2}\big) \to k^*\big(K_W^{\otimes^2}\big)$ of complex line bundles on X by the commutative diagram

$$
\begin{tikzcd}
j^*\big(K_V^{\otimes^2}\big) \arrow[r, "\cong"'] \arrow[d, "\Upsilon_{f,g}"] & \begin{array}{l}\big[\det(TX) \otimes j^*(K_V)\big] \otimes \big[k^*(K_W)^* \otimes \det(T^*X)\big] \otimes \\ \big[(\det(TX))^* \otimes j^*(K_V) \otimes k^*(K_W) \otimes (\det(T^*X))^*\big]\end{array} \arrow[d, "\alpha\otimes\beta^{-1}\otimes\mathrm{id}"'] \\
k^*\big(K_W^{\otimes^2}\big) & \begin{array}{l}\big[j^*(K_V)^* \otimes \det(T^*X)\big] \otimes \big[\det(TX) \otimes k^*(K_W)\big] \otimes \\ \big[(\det(TX))^* \otimes j^*(K_V) \otimes k^*(K_W) \otimes (\det(T^*X))^*\big].\end{array} \arrow[l, "\cong"']
\end{tikzcd}
\tag{6.3}
$$

These isomorphisms $\Upsilon_{f,g}$ are functorial, in that if $(U,e,i),(V,f,j),(W,g,k)$ are triples then

$$\Upsilon_{e,g}=\Upsilon_{f,g}\circ\Upsilon_{e,f},\quad \Upsilon_{g,f}=\Upsilon_{f,g}^{-1},\quad \textit{and}\quad \Upsilon_{f,f}=\mathrm{id}_{j^*(K_V^{\otimes^2})}. \tag{6.4}$$

(ii) *Let (V,f,j) and (W,g,k) be triples. Define a principal $\mathbb{Z}_2$-bundle $\pi_{f,g}:Q_{f,g}\to X$, such that on any open $Y\subseteq X$, continuous sections s of $Q_{f,g}|_Y\to Y$ correspond to isomorphisms $\iota_s:j^*(K_V)|_Y\to k^*(K_W)|_Y$ with $\iota_s\otimes\iota_s=\Upsilon_{f,g}|_Y: j^*\bigl(K_V^{\otimes^2}\bigr)|_Y\to k^*\bigl(K_W^{\otimes^2}\bigr)|_Y$. For any point $x\in X$, the fibre $\pi_{f,g}^{-1}(x)=\{\iota,-\iota\}$ is the set of isomorphisms $\iota:j^*(K_V)|_x\to k^*(K_W)|_x$ with $\iota\otimes\iota=\Upsilon_{f,g}|_x$.*

If $(U,e,i),(V,f,j),(W,g,k)$ are triples then there is a natural isomorphism $\Gamma_{e,f,g}:Q_{e,g}\to Q_{f,g}\otimes_{\mathbb{Z}_2}Q_{e,f}$ of principal $\mathbb{Z}_2$-bundles on X, such that if $Y\subseteq X$ is open and sections s,t,u of $Q_{e,f}|_Y,Q_{f,g}|_Y,Q_{e,g}|_Y$ correspond to isomorphisms $\iota_s:i^*(K_U)|_Y\to j^*(K_V)|_Y$, $\iota_t:j^*(K_V)|_Y\to k^*(K_W)|_Y$, $\iota_u:i^*(K_U)|_Y\to k^*(K_W)|_Y$ with $\iota_s\otimes\iota_s=\Upsilon_{e,f}|_Y$, $\iota_t\otimes\iota_t=\Upsilon_{f,g}|_Y$, $\iota_u\otimes\iota_u=\Upsilon_{e,g}|_Y$, then $\Gamma_{e,f,g}(u)=t\otimes_{\mathbb{Z}_2}s$ if and only if $\iota_u=\iota_t\circ\iota_s$.

These isomorphisms $\Gamma_{e,f,g}$ have the associativity property

$$\begin{aligned}\bigl(\mathrm{id}_{Q_{f,g}}\otimes\Gamma_{d,e,f}\bigr)\circ\Gamma_{d,f,g}=\bigl(\Gamma_{e,f,g}\otimes\mathrm{id}_{Q_{d,e}}\bigr)\circ\Gamma_{d,e,g}:&\\ Q_{d,g}\longrightarrow Q_{f,g}\otimes_{\mathbb{Z}_2}Q_{e,f}\otimes_{\mathbb{Z}_2}Q_{d,e}&\end{aligned}\tag{6.5}$$

for all triples $(T,d,h),(U,e,i),(V,f,j),(W,g,k)$.

Here is an example.

Example 6.2. Let $V=\mathbb{C}^*\times\mathbb{C}$, where $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, with coordinates (x,y) for $x\neq 0$. Let $k\in\mathbb{Z}$, and define $f,g:V\to\mathbb{C}$ by $f(x,y)=y^2$ and $g(x,y)=x^ky^2$. Then $X:=\mathrm{Crit}(f)=\mathrm{Crit}(g)=\mathbb{C}^*\times\{0\}\subset V$. Hence (V,f,id_X) and (V,g,id_X) are triples, as in Theorem 6.1. One can show that $\beta=x^{-k}\cdot\alpha$ in (6.2), and therefore $\Upsilon_{f,g}:K_V^{\otimes^2}|_X\to K_V^{\otimes^2}|_X$ in Theorem 6.1(i) is multiplication by x^k.

The principal $\mathbb{Z}_2$-bundle $P_{f,g}$ on $X\cong\mathbb{C}^*$ in Theorem 6.1(ii) parametrizes local holomorphic square roots of $x^k:\mathbb{C}^*\to\mathbb{C}^*$. As $H^1(\mathbb{C}^*;\mathbb{Z}_2)\cong\mathbb{Z}_2$ there are two principal $\mathbb{Z}_2$-bundles on $\mathbb{C}^*$ up to isomorphism, a trivial and a nontrivial. So $P_{f,g}$ is the trivial principal $\mathbb{Z}_2$-bundle if k is even (so that x^k has a global square root $x^{k/2}$ on $\mathbb{C}^*$, corresponding to a global section of $P_{f,g}\to X$), and the nontrivial principal $\mathbb{Z}_2$-bundle if k is odd (so that x^k has a no global square root, and $P_{f,g}$ no global section).

Remark 6.3. We can relate Theorem 6.1 to symmetric obstruction theories in Behrend [2]. Let X be a scheme, or complex analytic space. Then a *perfect obstruction theory* on X in the sense of Behrend and Fantechi [3] is a morphism $\phi:\mathcal{E}^\bullet\to\mathbb{L}_X$ in the derived category $D(\mathrm{qcoh}(X))$, where $\mathbb{L}_X$ is the cotangent complex of X, satisfying:

(i) $\mathcal{E}^\bullet$ is quasi-isomorphic locally on X to a complex $[\mathcal{F}^{-1}\to\mathcal{F}^0]$ of vector bundles in degrees $-1,0$;

(ii) $h^0(\phi):h^0(\mathcal{E}^\bullet)\to h^0(\mathbb{L}_X)$ is an isomorphism; and

(iii) $h^{-1}(\phi) : h^{-1}(\mathcal{E}^\bullet) \to h^{-1}(\mathbb{L}_X)$ is surjective.

We call $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ a *symmetric obstruction theory* if we also are given an isomorphism $\theta : \mathcal{E}^\bullet \to \mathcal{E}^{\bullet\vee}[1]$ with $\theta^\vee[1] = \theta$. As in Behrend [2] and Joyce and Song [11], moduli schemes of stable coherent sheaves and 'stable pairs' on Calabi–Yau 3-folds have symmetric obstruction theories. If V is a complex manifold and $f : V \to \mathbb{C}$ is holomorphic then $X = \mathrm{Crit}(f)$ has a natural symmetric obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ with

$$\mathcal{E}^\bullet = \big[TV|_X \xrightarrow{\partial^2 f|_X} T^*V|_X\big]. \tag{6.6}$$

Any perfect obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ has a *determinant line bundle* $\det \mathcal{E}^\bullet$, a holomorphic line bundle over X. For (6.6) we have $\det \mathcal{E}^\bullet = K_V^{\otimes^2}|_X$, so there is a natural square root line bundle $(\det \mathcal{E}^\bullet)^{1/2} = K_V|_X$.

Thus, we may interpret Theorem 6.1 as follows: the two triples (V, f, j) and (W, g, k) give us two symmetric obstruction theories $\phi_f : \mathcal{E}_f^\bullet \to \mathbb{L}_X$ and $\phi_g : \mathcal{E}_g^\bullet \to \mathbb{L}_X$ on X, with determinant line bundles $\det \mathcal{E}_f^\bullet = j^*\big(K_V^{\otimes^2}\big)$ and $\det \mathcal{E}_g^\bullet = k^*\big(K_W^{\otimes^2}\big)$. Theorem 6.1(i) gives a canonical isomorphism $\Upsilon_{f,g} : \det \mathcal{E}_f^\bullet \to \det \mathcal{E}_g^\bullet$, and the principal $\mathbb{Z}_2$-bundle $P_{f,g} \to X$ parametrizes isomorphisms $\iota : (\det \mathcal{E}_f^\bullet)^{1/2} \to (\det \mathcal{E}_g^\bullet)^{1/2}$ with $\iota \otimes \iota = \Upsilon_{f,g}$.

All this will be important in the sequel [6]. Loosely following Kontsevich and Soibelman [14, §5], we will define *orientation data* for a scheme with perfect obstruction theory X, $\phi : \mathcal{E}^\bullet \to \mathbb{L}_X$ to be a choice of square root line bundle $(\det \mathcal{E}^\bullet)^{1/2}$ for $\det \mathcal{E}^\bullet$, if such a square root exists. Thus, $P_{f,g}$ relates the natural choices of orientation data on $\mathrm{Crit}(f)$ and $\mathrm{Crit}(g)$.

In [6] we will use Theorems 6.1, 6.4 and 6.7 to show that if $\boldsymbol{X}$ is a quasi-smooth derived $\mathbb{C}$-scheme with -1-shifted symplectic structure ω in the sense of Pantev, Toën, Vaquié and Vezzosi [20], with a choice of orientation data, then we can construct a perverse sheaf $\mathcal{P}^\bullet_{\boldsymbol{X},\omega}$ on $t_0(\boldsymbol{X})$, which has important applications in the 'categorification' of Donaldson–Thomas theory on Calabi–Yau 3-folds.

The second part of our main result, proved in §6.2–§6.4, studies a notion of 'compatibility' for triples $(V, f, j), (W, g, k)$. The initial definition is weak: $(V, f, j), (W, g, k)$ are compatible if $g \circ \Phi - f \in I_{\mathrm{d}f}^2$ for local $\Phi : V \to W$ with $\Phi \circ j = k$. But this is equivalent to an apparently much stronger condition, that we may locally identify $W \cong V \times \mathbb{C}^n$ so that $g \cong f \boxplus z_1^2 + \cdots + z_n^2$.

Theorem 6.4. *Let X be a fixed complex analytic space, and* ***triples*** (V, f, j) *be as in Theorem* 6.1.

(a) *Let (V, f, j) and (W, g, k) be triples. For each $x \in X$, there exists an open neighbourhood V' of $j(x)$ in V, so that $X' := j^{-1}(V')$ is an open neighbourhood of x in X, and a holomorphic map $\Phi : V' \to W$ such that $\Phi \circ j|_{X'} = k|_{X'}$ as a morphism of complex analytic spaces $X' \to W$.*

Write $\mathcal{O}_V$ for the complex analytic sheaf of holomorphic functions on V, and $I_{\mathrm{d}f} \subseteq \mathcal{O}_V$ for the ideal generated by $\mathrm{d}f$, or equivalently, the ideal of holomorphic functions in $\mathcal{O}_V$ vanishing on $j(X) = \mathrm{Crit}(f)$. Then $I_{\mathrm{d}f}^2$ is also an ideal in $\mathcal{O}_V$, and we can form the quotient sheaf $\mathcal{O}_V / I_{\mathrm{d}f}^2$.

For x, V', Φ as above, we have a holomorphic function $g \circ \Phi : V' \to \mathbb{C}$, so $g \circ \Phi + I_{\mathrm{d}f}^2|_{V'} \in H^0\big((\mathcal{O}_V/I_{\mathrm{d}f}^2)|_{V'}\big)$. This $g \circ \Phi + I_{\mathrm{d}f}^2|_{V'}$ is independent of the choice of V', Φ near x, in the sense that if $\tilde{V}', \tilde{\Phi}$ are alternative choices for V', Φ then in $H^0\big((\mathcal{O}_V/I_{\mathrm{d}f}^2)|_{V'\cap\tilde{V}'}\big)$ we have

$$g \circ \Phi|_{V'\cap\tilde{V}'} + I_{\mathrm{d}f}^2|_{V'\cap\tilde{V}'} = g \circ \tilde{\Phi}|_{V'\cap\tilde{V}'} + I_{\mathrm{d}f}^2|_{V'\cap\tilde{V}'}. \tag{6.7}$$

*Define two triples $(V, f, j), (W, g, k)$ to be **compatible** if for all x, V', Φ as above we have*

$$(f + I_{\mathrm{d}f}^2)|_{V'} = g \circ \Phi + I_{\mathrm{d}f}^2|_{V'} \quad \text{in } H^0\big((\mathcal{O}_V/I_{\mathrm{d}f}^2)|_{V'}\big). \tag{6.8}$$

This condition is independent of the choice of V', Φ near x by (6.7)*. Compatibility is an equivalence relation on triples (V, f, j).*

(b) *Suppose (V, f, j) and (W, g, k) are compatible triples, with $\dim V \leqslant \dim W$, and write $n = \dim W - \dim V$. Then for each $x \in X$ there exists an open neighbourhood W' of $k(x)$ in W, so that $X' = k^{-1}(W')$ is an open neighbourhood of x in X, and a holomorphic map $\Psi : W' \to V \times \mathbb{C}^n$ which is a biholomorphism with an open neighbourhood $\Psi(W')$ of $(j(x), 0)$ in $V \times \mathbb{C}^n$, such that $\Psi \circ k|_{X'} = (j \times 0)|_{X'}$ as morphisms of complex analytic spaces $X' \to V \times \mathbb{C}^n$, and if $w \in W'$ with $\Psi(w) = \big(v, (z_1, \ldots, z_n)\big)$ then $g(w) = f(v) + z_1^2 + \cdots + z_n^2$.*

Conversely, if $(V, f, j), (W, g, k)$ are triples with $\dim V \leqslant \dim W$ and such W', Ψ exist for all $x \in X$, then $(V, f, j), (W, g, k)$ are compatible.

(c) *We can relate part* **(b)** *to Theorem* 6.1 *as follows. For $(V, f, j), (W, g, k)$ and x, W', Ψ as in* **(b)**, *$\mathrm{d}\Psi : TW|_{W'} \to \Psi^*(TV \oplus T\mathbb{C}^n)$ is an isomorphism of vector bundles over W'. Thus we have isomorphisms*

$$\begin{aligned} k|_{X'}^*(\mathrm{d}\Psi^*) &: \big(j^*(T^*V) \oplus T_0^*\mathbb{C}^n)\big)|_{X'} \longrightarrow k^*(T^*W)|_{X'}, \\ \Lambda^{\dim W}\big(k|_{X'}^*(\mathrm{d}\Psi^*)\big) &: j^*(K_V)|_{X'} \otimes \Lambda^n(T_0^*\mathbb{C}^n) \longrightarrow k^*(K_W)|_{X'}. \end{aligned}$$

Define $\iota_s : j^*(K_V)|_{X'} \xrightarrow{\cong} k^*(K_W)|_{X'}$ by the commutative diagram

$$\begin{tikzcd} j^*(K_V)|_{X'} \arrow[d, "\mathrm{id}\otimes(\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n)"] \arrow[dr, "\iota_s"] & \\ j^*(K_V)|_{X'} \otimes \Lambda^n(T_0^*\mathbb{C}^n) \arrow[r, "\Lambda^{\dim W}(k|_{X'}^*(\mathrm{d}\Psi^*))"] & k^*(K_W)|_{X'}. \end{tikzcd} \tag{6.9}$$

Then using the notation of Theorem 6.1*, we have*

$$\iota_s \otimes \iota_s = \Upsilon_{f,g}|_{X'} : j^*\big(K_V^{\otimes^2}\big)|_{X'} \longrightarrow k^*\big(K_W^{\otimes^2}\big)|_{X'}. \tag{6.10}$$

Thus ι_s corresponds to a unique section s of $Q_{f,g}|_{X'}$ by Theorem 6.1(ii)*, so $Q_{f,g}|_{X'}$ is a trivial principal $\mathbb{Z}_2$-bundle over X'.*

Note the similarity of Theorem 6.4(b) and Theorem 5.1(i).

Example 6.5. Let V be an open neighbourhood of 0 in $\mathbb{C}$, let $3 \leqslant n \leqslant N$, and define polynomial holomorphic functions $f, g : V \to \mathbb{C}$ by

$$f(z) = a_n z^n + \cdots + a_N z^N, \quad g(z) = b_n z^n + \cdots + b_N z^N, \tag{6.11}$$

for $a_j, b_j \in \mathbb{C}$ for $j = n, \ldots, N$ with $a_n, b_n \neq 0$. Then 0 is an isolated critical point of f, g, so making V smaller if necessary we can suppose 0 is the only critical point of f, g in V.

Then $X = \mathrm{Crit}(f) = \mathrm{Crit}(g)$ is the complex analytic subspace $z^{n-1} = 0$ in $V \subseteq \mathbb{C}$. So $(V, f, \mathrm{id}_X), (V, g, \mathrm{id}_X)$ are triples as in Theorem 6.1. Using $\Phi_0 = \mathrm{id}_V$ in Theorem 6.4(a), we see that $(V, f, \mathrm{id}_X), (V, g, \mathrm{id}_X)$ are compatible if and only if $f(z) + (z^{n-1})^2 = g(z) + (z^{n-1})^2$. Thus from (6.11) we see that (V, f, id_X) and (V, g, id_X) are compatible if and only if $a_j = b_j$ for $j = n, \ldots, 2n-3$.

Remark 6.6. **(i)** The notion of compatible triples $(V, f, j), (W, g, k)$ in Theorem 6.4 is quite subtle. The *only* relation between (V, f) and (W, g) that we use is the isomorphism $k \circ j^{-1} : \mathrm{Crit}(f) \to \mathrm{Crit}(g)$ of complex analytic spaces. Since functions on $\mathrm{Crit}(f)$ are $\mathcal{O}_V / I_{\mathrm{d}f}$, one might expect that an isomorphism $k \circ j^{-1} : \mathrm{Crit}(f) \to \mathrm{Crit}(g)$ could only determine functions modulo $I_{\mathrm{d}f}$. But in (6.8) we can compare f, g modulo $I_{\mathrm{d}f}^2$ in a well-defined way.

(ii) In Example 6.5, $(V, f, \mathrm{id}_X), (V, g, \mathrm{id}_X)$ are compatible if $a_j = b_j$ for $j = n, \ldots, 2n-3$. As in Remark 6.3, the symmetric obstruction theory (6.6) on $X = \mathrm{Crit}(f)$ determined by f depends on $\partial^2 f|_X = n(n-1)a_n z^{n-2} + \cdots$ modulo (z^{n-1}). Thus, in Example 6.5, $\mathrm{Crit}(f), \mathrm{Crit}(g)$ are equivalent as schemes with symmetric obstruction theories if $a_n = b_n$.

We will show in [6] that $(V, f, j), (W, g, k)$ are compatible if $\mathrm{Crit}(f), \mathrm{Crit}(g)$ are equivalent as quasi-smooth derived $\mathbb{C}$-schemes with -1-shifted symplectic structures, in the sense of [20]. The classical $\mathbb{C}$-scheme with symmetric obstruction theory is a truncation of this, containing less information. In Example 6.5, the 'derived' structure remembers the $n-2$ coefficients $a_n, \ldots, a_{2n-3}$ in f, but the classical truncation remembers only the one coefficient a_n.

Here is the final part of our main result, proved in §6.5–§6.7.

Theorem 6.7. *Let X be a fixed complex analytic space, and use all the notation of Theorems* 6.1 *and* 6.4.

(i) *Let (V, f, j) and (W, g, k) be compatible triples, and $Q_{f,g}$ be as in Theorem* 6.1(ii). *Then there exists a natural isomorphism of perverse sheaves on X:*

$$\Delta_{f,g} : j^*(\mathcal{PV}_{V,f}^\bullet) \longrightarrow k^*(\mathcal{PV}_{W,g}^\bullet) \otimes_{\mathbb{Z}_2} Q_{f,g}. \tag{6.12}$$

Also the following diagram commutes, where $\tau_{V,f}, \tau_{W,g}$ are the twisted monodromy operators from (2.7):

$$\begin{tikzcd} j^*(\mathcal{PV}_{V,f}^\bullet) \arrow[r, "\Delta_{f,g}"'] \arrow[d, "j^*(\tau_{V,f})"] & k^*(\mathcal{PV}_{W,g}^\bullet) \otimes_{\mathbb{Z}_2} Q_{f,g} \arrow[d, "k^*(\tau_{W,g}) \otimes \mathrm{id}_{Q_{f,g}}"'] \\ j^*(\mathcal{PV}_{V,f}^\bullet) \arrow[r, "\Delta_{f,g}"] & k^*(\mathcal{PV}_{W,g}^\bullet) \otimes_{\mathbb{Z}_2} Q_{f,g}. \end{tikzcd} \tag{6.13}$$

(ii) *If* $(U,e,i),(V,f,j),(W,g,k)$ *are compatible triples and* $\Gamma_{e,f,g}$ *is as in Theorem* 6.1(ii), *then we have a commutative diagram of isomorphisms in* $\mathrm{Perv}(X)$:

$$\begin{tikzcd} i^*(\mathcal{PV}^\bullet_{U,e}) \arrow[rr, "\Delta_{e,f}"'] \arrow[d, "\Delta_{e,g}"] & & j^*(\mathcal{PV}^\bullet_{V,f}) \otimes_{\mathbb{Z}_2} Q_{e,f} \arrow[d, "\Delta_{f,g}\otimes \mathrm{id}_{Q_{e,f}}"'] \\ k^*(\mathcal{PV}^\bullet_{W,g}) \otimes_{\mathbb{Z}_2} Q_{e,g} \arrow[rr, "\mathrm{id}\otimes\Gamma_{e,f,g}"] & & k^*(\mathcal{PV}^\bullet_{W,g}) \otimes_{\mathbb{Z}_2} Q_{f,g} \otimes_{\mathbb{Z}_2} Q_{e,f}. \end{tikzcd} \tag{6.14}$$

(iii) *The analogues of* **(i)**,**(ii)** *hold for mixed Hodge modules. That is, for compatible triples* $(V,f,j),(W,g,k)$ *we have a natural isomorphism*

$$\Delta^H_{f,g} : j^*(\mathcal{HV}^\bullet_{V,f}) \longrightarrow k^*(\mathcal{HV}^\bullet_{W,g}) \otimes_{\mathbb{Z}_2} Q_{f,g}. \tag{6.15}$$

in $\mathrm{MHM}(X;T_s,N)$. *Also, for compatible* $(U,e,i),(V,f,j),(W,g,k)$ *the following commutes:*

$$\begin{tikzcd} i^*(\mathcal{HV}^\bullet_{U,e}) \arrow[rr, "\Delta^H_{e,f}"'] \arrow[d, "\Delta^H_{e,g}"] & & j^*(\mathcal{HV}^\bullet_{V,f}) \otimes_{\mathbb{Z}_2} Q_{e,f} \arrow[d, "\Delta^H_{f,g}\otimes \mathrm{id}_{Q_{e,f}}"'] \\ k^*(\mathcal{HV}^\bullet_{W,g}) \otimes_{\mathbb{Z}_2} Q_{e,g} \arrow[rr, "\mathrm{id}\otimes\Gamma_{e,f,g}"] & & k^*(\mathcal{HV}^\bullet_{W,g}) \otimes_{\mathbb{Z}_2} Q_{f,g} \otimes_{\mathbb{Z}_2} Q_{e,f}. \end{tikzcd} \tag{6.16}$$

The proof of Theorem 6.7 is similar to that of Theorem 5.2.

Remark 6.8. **(a)** Theorems 4.1 and 5.1–5.2 are both in effect special cases of Theorems 6.1, 6.4 and 6.7, in the following ways.

In Theorem 4.1, making V smaller so that $X=\mathrm{Crit}(f)=\mathrm{Crit}(g)$, we find that $(V,f,\mathrm{id}_X),(V,g,\mathrm{id}_X)$ are compatible triples, $\Upsilon_{f,g}=\mathrm{id}_{K_V^{\otimes^2}|_X}$, and $Q_{f,g}$ is naturally isomorphic to the trivial $\mathbb{Z}_2$-bundle over X, and $\Lambda_{f,g}$ in (4.2) coincides with $\Delta_{f,g}$ in (6.12) under the trivialization of $Q_{f,g}$.

Similarly, in Theorems 5.1–5.2, $(V,f,\mathrm{id}_X),(W,g,\mathrm{id}_X)$ are compatible triples, and $P_{f,g}$ in Theorems 5.1–5.2 is naturally isomorphic to $Q_{f,g}$ in Theorem 6.1(ii), and W',Ψ in Theorems 5.1–5.2 coincide with W',Ψ in Theorem 6.7(b), and $\Theta_{f,g}$ in (5.2) coincides with $\Delta_{f,g}$ in (6.12) under $P_{f,g}\cong Q_{f,g}$.

The proofs of Theorems 6.4 and 6.7 depend on those of Theorems 4.1 and 5.1–5.2. Essentially, we use the method of Theorem 4.1 to prove Theorem 6.4(b), and then deduce Theorem 6.7 from Theorems 5.1–5.2.

(b) It is surprising that given $f:V\to\mathbb{C}$ and $g:W\to\mathbb{C}$, the *only* data required to define the canonical isomorphism $\Delta_{f,g}$ in (6.12) relating $\mathcal{PV}^\bullet_{V,f}$ and $\mathcal{PV}^\bullet_{W,g}$ is the isomorphism of complex analytic spaces $k\circ j^{-1}:\mathrm{Crit}(f)\to\mathrm{Crit}(g)$, though $k\circ j^{-1}$ must satisfy compatibility in Theorem 6.4.

One might have expected that defining $\Delta_{f,g}$ would require more 'derived' data, for instance, an equivalence of the symmetric obstruction theories on $\mathrm{Crit}(f),\mathrm{Crit}(g)$, or even an equivalence of the full quasi-smooth derived $\mathbb{C}$-schemes with -1-shifted symplectic structures in Pantev, Toën, Vaquié and Vezzosi [20], as discussed in Remarks 6.3 and 6.6. But an isomorphism at the level of classical schemes or complex analytic spaces is enough to define $\Delta_{f,g}$.

The rest of the section proves Theorems 6.1, 6.4 and 6.7.

6.1 The proof of Theorem 6.1

Part (i) up to (6.3) is definitions, and needs no proof. For (6.4), suppose $(U,e,i),(V,f,j),(W,g,k)$ are triples. Write α,β,γ for the isomorphisms

$$\begin{aligned}
\alpha &: \det(TX)\otimes i^*(K_U) \longrightarrow i^*(K_U)^*\otimes \det(T^*X),\\
\beta &: \det(TX)\otimes j^*(K_V) \longrightarrow j^*(K_V)^*\otimes \det(T^*X),\\
\gamma &: \det(TX)\otimes k^*(K_W) \longrightarrow k^*(K_W)^*\otimes \det(T^*X),
\end{aligned}$$

induced by (6.1) for $(U,e,i),\ldots,(W,g,k)$, as in (6.2). Consider the diagram:

$$\begin{tikzcd}
i^*\bigl(K_U^{\otimes^2}\bigr) \arrow[rr, "\cong"] \arrow[dr, "\cong"] \arrow[ddd, "\Upsilon_{e,f}"] \arrow[dddddd, bend right, "\Upsilon_{e,g}"'] & & \begin{array}{l}\bigl[\det(TX)\otimes i^*(K_U)\bigr]\otimes\bigl[k^*(K_W)^*\otimes\det(T^*X)\bigr]\otimes\\ \bigl[(\det(TX))^*\otimes i^*(K_U)\otimes k^*(K_W)\otimes(\det(T^*X))^*\bigr]\end{array} \arrow[dl, "\cong"'] \arrow[dddddd, "\alpha\otimes\gamma^{-1}\otimes\mathrm{id}"] \\
& \begin{array}{l}\bigl[\det(TX)\otimes i^*(K_U)\bigr]\otimes\bigl[j^*(K_V)^*\otimes\det(T^*X)\bigr]\otimes\\ \bigl[(\det(TX))^*\otimes i^*(K_U)\otimes j^*(K_V)\otimes(\det(T^*X))^*\bigr]\end{array} \arrow[d, "\alpha\otimes\beta^{-1}\otimes\mathrm{id}"'] & \\
& \begin{array}{l}\bigl[i^*(K_U)^*\otimes\det(T^*X)\bigr]\otimes\bigl[\det(TX)\otimes j^*(K_V)\bigr]\otimes\\ \bigl[(\det(TX))^*\otimes i^*(K_U)\otimes j^*(K_V)\otimes(\det(T^*X))^*\bigr]\end{array} \arrow[dl, "\cong"'] \arrow[dd, "\cong"'] & \\
j^*\bigl(K_V^{\otimes^2}\bigr) \arrow[dr, "\cong"] \arrow[ddd, "\Upsilon_{f,g}"] & & \\
& \begin{array}{l}\bigl[\det(TX)\otimes j^*(K_V)\bigr]\otimes\bigl[k^*(K_W)^*\otimes\det(T^*X)\bigr]\otimes\\ \bigl[(\det(TX))^*\otimes j^*(K_V)\otimes k^*(K_W)\otimes(\det(T^*X))^*\bigr]\end{array} \arrow[d, "\beta\otimes\gamma^{-1}\otimes\mathrm{id}"'] & \\
& \begin{array}{l}\bigl[j^*(K_V)^*\otimes\det(T^*X)\bigr]\otimes\bigl[\det(TX)\otimes k^*(K_W)\bigr]\otimes\\ \bigl[(\det(TX))^*\otimes j^*(K_V)\otimes k^*(K_W)\otimes(\det(T^*X))^*\bigr]\end{array} \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
k^*\bigl(K_W^{\otimes^2}\bigr) & & \begin{array}{l}\bigl[i^*(K_U)^*\otimes\det(T^*X)\bigr]\otimes\bigl[\det(TX)\otimes k^*(K_W)\bigr]\otimes\\ \bigl[(\det(TX))^*\otimes i^*(K_U)\otimes k^*(K_W)\otimes(\det(T^*X))^*\bigr].\end{array} \arrow[ll, "\cong"']
\end{tikzcd}$$

Here the top left quadrilateral, bottom left quadrilateral, and outer rectangle are (6.3) for $\Upsilon_{e,f},\Upsilon_{f,g},\Upsilon_{e,g}$ respectively, and so commute. The right hand hexagon commutes as $\alpha\otimes\gamma^{-1}=(\beta\otimes\gamma^{-1})\circ(\alpha\otimes\beta^{-1})$, omitting canonical isomorphisms. Thus the whole diagram commutes, and $\Upsilon_{e,g}=\Upsilon_{f,g}\circ\Upsilon_{e,f}$, proving the first equation of (6.4).

Applying this to $(V,f,j),(V,f,j),(W,g,k)$ gives $\Upsilon_{f,g}=\Upsilon_{f,g}\circ\Upsilon_{f,f}$, so $\Upsilon_{f,f}=\mathrm{id}_{j^*(K_V^{\otimes^2})}$ as $\Upsilon_{f,g}$ is invertible. Applying it to $(V,f,j),(W,g,k),(V,f,j)$ gives $\Upsilon_{f,f}=\Upsilon_{g,f}\circ\Upsilon_{f,g}$, so $\Upsilon_{g,f}=\Upsilon_{f,g}^{-1}$ as $\Upsilon_{f,f}=\mathrm{id}_{j^*(K_V^{\otimes^2})}$. This proves the last two equations of (6.4), and part (i) of Theorem 6.1.

For part (ii), the definitions of $\pi_{f,g}:Q_{f,g}\to X$ and of $\Gamma_{e,f,g}$ with the given properties are essentially immediate. To prove (6.5), let $(T,d,h),(U,e,i),(V,f,j),(W,g,k)$ be triples, $Y\subseteq X$ be open, and $\iota_s:h^*(K_T)|_Y\to i^*(K_U)|_Y$, $\iota_t:i^*(K_U)|_Y\to j^*(K_V)|_Y$, $\iota_u:j^*(K_V)|_Y\to k^*(K_W)|_Y$ be isomorphisms with $\iota_s\otimes\iota_s=\Upsilon_{d,e}|_Y$, $\iota_t\otimes\iota_t=\Upsilon_{e,f}|_Y$ and $\iota_u\otimes\iota_u=\Upsilon_{f,g}|_Y$. Then ι_s,ι_t,ι_u correspond to sections s,t,u of $Q_{d,e}|_Y,Q_{e,f}|_Y,Q_{f,g}|_Y$, respectively.

The definition of $\Gamma_{e,f,g}$ now implies that we have correspondences between sections of principal $\mathbb{Z}_2$-bundles and isomorphisms of line bundles over Y:

$$\Gamma_{d,e,f}|_Y^{-1}(t\otimes s)\longleftrightarrow \iota_t\circ\iota_s,\quad \Gamma_{d,f,g}|_Y^{-1}\bigl(u\otimes\Gamma_{d,e,f}|_Y^{-1}(t\otimes s)\bigr)\longleftrightarrow \iota_u\circ(\iota_t\circ\iota_s),$$
$$\Gamma_{e,f,g}|_Y^{-1}(u\otimes t)\longleftrightarrow \iota_u\circ\iota_t,\quad \Gamma_{d,e,g}|_Y^{-1}\bigl(\Gamma_{e,f,g}|_Y^{-1}(u\otimes t)\otimes s\bigr)\longleftrightarrow (\iota_u\circ\iota_t)\circ\iota_s.$$

Since $\iota_u\circ(\iota_t\circ\iota_s)=(\iota_u\circ\iota_t)\circ\iota_s$, this shows that

$$\begin{aligned}\bigl[\bigl(\mathrm{id}_{Q_{f,g}}\otimes\Gamma_{d,e,f}\bigr)\circ\Gamma_{d,f,g}\bigr]\big|_Y^{-1}(u\otimes t\otimes s)&=\Gamma_{d,f,g}|_Y^{-1}\bigl(u\otimes\Gamma_{d,e,f}|_Y^{-1}(t\otimes s)\bigr)\\&=\Gamma_{d,e,g}|_Y^{-1}\bigl(\Gamma_{e,f,g}|_Y^{-1}(u\otimes t)\otimes s\bigr)\\&=\bigl[\bigl(\Gamma_{e,f,g}\otimes\mathrm{id}_{Q_{d,e}}\bigr)\circ\Gamma_{d,e,g}\bigr]\big|_Y^{-1}(u\otimes t\otimes s).\end{aligned}$$

Hence the restriction of (6.5) to Y holds. Since we can cover X by small open subspaces Y on which such square roots ι_s,ι_t,ι_u for $\Upsilon_{d,e},\Upsilon_{e,f},\Upsilon_{f,g}$ exist, (6.5) holds on all of X. This completes the proof of Theorem 6.1.

6.2 Theorem 6.4(a): compatible triples $(V,f,j),(W,g,k)$

Let $(V,f,j),(W,g,k)$ be triples, and $x\in X$. Choose holomorphic coordinates $(z_1,\ldots,z_m)$ defined on an open neighbourhood W' of $k(x)$ in W, where $m=\dim W$. Then $z_a\circ k\circ j^{-1}\in H^0\bigl((\mathcal{O}_V/I_{\mathrm{d}f})|_{j\circ k^{-1}(W')}\bigr)$ for each $a=1,\ldots,m$. Now locally near $j(x)$, sections of $\mathcal{O}_V/I_{\mathrm{d}f}$ lift (non-uniquely) to sections of $\mathcal{O}_V$. Thus we can choose an open neighbourhood V' of $j(x)$ in V and holomorphic functions $w_1,\ldots,w_m\in H^0\bigl(\mathcal{O}_V|_{V'}\bigr)$ such that for $a=1,\ldots,m$ we have

$$(w_a+I_{\mathrm{d}f})|_{V'\cap j\circ k^{-1}(W')}=z_a\circ k\circ j^{-1}|_{V'\cap j\circ k^{-1}(W')}. \tag{6.17}$$

We now have holomorphic maps $(w_1,\ldots,w_m):V'\to\mathbb{C}^m$ and $(z_1,\ldots,z_m):W'\to\mathbb{C}^m$, with $(z_1,\ldots,z_m)$ a biholomorphism with its image. By (6.17), writing $X'=j^{-1}(V')\cap k^{-1}(W')$, as an open neighbourhood of x in X, we have

$$(w_1,\ldots,w_m)\circ j|_{X'}=(z_1,\ldots,z_m)\circ k|_{X'}:X'\longrightarrow\mathbb{C}^m. \tag{6.18}$$

Making V' smaller, we can suppose that $j^{-1}(V')\subseteq k^{-1}(W')$, so that $X'=j^{-1}(V')$, and $(w_1,\ldots,w_m)(V')\subseteq(z_1,\ldots,z_m)(W')\subseteq\mathbb{C}^m$. Define $\Phi:V'\to W$ by $\Phi=(z_1,\ldots,z_m)^{-1}\circ(w_1,\ldots,w_m)$. Then (6.18) implies that $\Phi\circ j|_{X'}=k|_{X'}:X'\to W$. This proves the first part of Theorem 6.4(a).

For the second part, let $x\in X$, and V',Φ and $\tilde V',\tilde\Phi$ satisfy the conditions of the first part. Let $y\in\mathrm{Crit}(f)\cap V'\cap\tilde V'$, and choose local holomorphic coordinates $(z_1,\ldots,z_m)$ on W near $\Phi(y)=\tilde\Phi(y)$. Then near y in $V'\cap\tilde V'$, by a holomorphic version of Taylor's Theorem we have

$$\begin{aligned}g\circ\tilde\Phi-g\circ\Phi=&\textstyle\sum_{a=1}^m\bigl(\frac{\partial g}{\partial z_a}\circ\Phi\bigr)\cdot\bigl(z_a\circ\tilde\Phi-z_a\circ\Phi\bigr)\\&+\textstyle\sum_{a,b=1}^m A_{ab}\bigl(z_a\circ\tilde\Phi-z_a\circ\Phi\bigr)\bigl(z_b\circ\tilde\Phi-z_b\circ\Phi\bigr),\end{aligned} \tag{6.19}$$

for some holomorphic $A_{ab}:V\to\mathbb{C}$ defined near y.

Since $\tilde{\Phi}\circ j|_{X'}=\Phi\circ j|_{X'}=k|_{X'}$ and $j(X')$ is open in $\mathrm{Crit}(f)=\mathrm{d}f^{-1}(0)$, we see that $z_a\circ\tilde{\Phi}-z_a\circ\Phi\in I_{\mathrm{d}f}$ near y for each $a=1,\dots,m$. Also $\frac{\partial g}{\partial z_a}\in I_{\mathrm{d}g}$, and Φ maps $\mathrm{d}f^{-1}(0)\to\mathrm{d}g^{-1}(0)$, so $\frac{\partial g}{\partial z_a}\circ\Phi\in I_{\mathrm{d}f}$ near y. Thus each factor $(\cdots)$ on the right hand side of (6.19) lies in $I_{\mathrm{d}f}$ near y, so $g\circ\tilde{\Phi}-g\circ\Phi\in I_{\mathrm{d}f}^2$ near y. As this holds for each $y\in\mathrm{Crit}(f)\cap V'\cap\tilde{V}'$, equation (6.7) follows.

We say that (V,f,j) *is compatible with* (W,g,k), or $(V,f,j)\sim(W,g,k)$ for short, if (6.8) holds for all x,V',Φ as above. This is independent of the choice of V',Φ near x by (6.7). We will show $\sim$ is an equivalence relation.

Suppose $(V,f,j)\sim(W,g,k)$. Let $x\in X$ and V',Φ be as above, so that (6.8) holds. Let W' be an open neighbourhood of $k(x)$ in W, so that $X''=k^{-1}(W')$ is an open neighbourhood of x in X, and $\Psi:W'\to V$ be holomorphic with $\Psi\circ k|_{X''}=j|_{X''}$. The first part of Theorem 6.4(a) with $(V,f,j),(W,g,k)$ exchanged shows such W',Ψ' exist. Making W' smaller we can suppose $\Psi'(W')\subseteq V'$. As Ψ maps $k(X)=\mathrm{d}g^{-1}(0)\to j(X)=\mathrm{d}f^{-1}(0)$, pullback Ψ^* maps $I_{\mathrm{d}f}\to I_{\mathrm{d}g}$, and thus maps $I_{\mathrm{d}f}^2\to I_{\mathrm{d}g}^2$. Hence composing (6.8) with Ψ gives

$$f\circ\Psi+I_{\mathrm{d}g}^2|_{W'}=g\circ(\Phi\circ\Psi)+I_{\mathrm{d}g}^2|_{W'}. \tag{6.20}$$

Equation (6.7) with $(W,g,k),W',\Phi\circ\Psi,W',\mathrm{id}_{W'}$ in place of $(V,f,j),V',\Phi,\tilde{V}',\tilde{\Phi}$ yields

$$g\circ(\Phi\circ\Psi)+I_{\mathrm{d}g}^2|_{W'}=g\circ\mathrm{id}_{W'}+I_{\mathrm{d}g}^2|_{W'}. \tag{6.21}$$

Combining (6.20)–(6.21) proves (6.8) with $(V,f,j),(W,g,k)$ exchanged. Hence $(W,g,k)\sim(V,f,j)$, and $\sim$ is symmetric.

Suppose $(U,e,i)\sim(V,f,j)$ and $(V,f,j)\sim(W,g,k)$. Let $x\in X$. Then there exist open $i(x)\in U'\subseteq U$, $j(x)\in V'\subseteq V$ and holomorphic $\Phi:U'\to V$, $\Psi:V'\to W$ such that $\Phi\circ i|_{i^{-1}(U')}=j|_{i^{-1}(U')}$ and $\Psi\circ j|_{j^{-1}(V')}=k|_{j^{-1}(V')}$, and equation (6.8) yields

$$(e+I_{\mathrm{d}e}^2)|_{U'}=f\circ\Phi+I_{\mathrm{d}e}^2|_{U'}, \tag{6.22}$$
$$(f+I_{\mathrm{d}f}^2)|_{V'}=g\circ\Psi+I_{\mathrm{d}f}^2|_{V'}. \tag{6.23}$$

Set $\tilde{U}'=\Psi^{-1}(V')\subseteq U'$ and $\tilde{\Phi}=\Psi\circ\Phi|_{\tilde{U}'}:\tilde{U}'\to W$. Since Φ^* maps $I_{\mathrm{d}f}\to I_{\mathrm{d}e}$, and thus maps $I_{\mathrm{d}f}^2\to I_{\mathrm{d}e}^2$, composing (6.23) with $\Phi|_{\tilde{U}'}$ gives

$$f\circ\Phi|_{\tilde{U}'}+I_{\mathrm{d}e}^2|_{\tilde{U}'}=g\circ\Psi\circ\Phi|_{\tilde{U}'}+I_{\mathrm{d}e}^2|_{\tilde{U}'}=g\circ\tilde{\Phi}+I_{\mathrm{d}e}^2|_{\tilde{U}'}.$$

Combining this with the restriction of (6.22) to $\tilde{U}'$ gives

$$(e+I_{\mathrm{d}e}^2)|_{\tilde{U}'}=g\circ\tilde{\Phi}+I_{\mathrm{d}e}^2|_{\tilde{U}'}.$$

This is equation (6.8) for $(U,e,i),(W,g,k)$ and $x,\tilde{U},\tilde{\Phi}$. As the condition is independent of the choice of $\tilde{U},\tilde{\Phi}$ near x, we see that $(U,e,i)\sim(W,g,k)$, and $\sim$ is transitive. Thus, compatibility $\sim$ is an equivalence relation on triples (V,f,j), completing the proof of Theorem 6.4(a).

6.3 Theorem 6.4(b): finding Ψ with $g = (f \boxplus z_1^2 + \cdots + z_n^2) \circ \Psi$

Let (V, f, j) and (W, g, k) be compatible triples, with $n = \dim W - \dim V \geqslant 0$, and let $x \in X$. The proof of Proposition 3.3 in §3.2 shows that we may choose a complex submanifold P of V with $j(x) \in P$, such that $T_{j(x)}P = T_{j(x)} \operatorname{Crit}(f) \subseteq T_{j(x)}V$, and setting $r = f|_P : P \to \mathbb{C}$, then $\operatorname{Crit}(r)$ is an open neighbourhood of $j(x)$ in $\operatorname{Crit}(f)$, considered as complex analytic subspaces of V.

Let V', Φ be as in Theorem 6.4(a), so that (6.8) holds as $(V, f, j), (W, g, k)$ are compatible. Since Φ maps $\operatorname{Crit}(f) \xrightarrow{\cong} \operatorname{Crit}(g)$ locally, $\mathrm{d}\Phi|_{j(x)}$ induces an isomorphism from $T_{j(x)}P = T_{j(x)} \operatorname{Crit}(f)$ to $T_{k(x)} \operatorname{Crit}(g) \subseteq T_{k(x)}W$. Therefore $\Phi|_{P \cap V'}$ is an embedding near $j(x)$. Making P smaller, we can suppose that $P \subseteq V'$ and $\Phi|_P : P \to W$ is an embedding. Thus $Q = \Phi(P)$ is a complex submanifold of W, which contains an open neighbourhood of $k(x)$ in $\operatorname{Crit}(g)$ as a complex analytic subspace, and $\Phi|_P : P \to Q$ is a biholomorphism. Write $s = g|_Q : Q \to \mathbb{C}$. Then restricting (6.8) to P yields

$$r + I_{\mathrm{d}r}^2 = s \circ \Phi|_P + I_{\mathrm{d}r}^2 \quad \text{in } H^0\big(\mathcal{O}_P / I_{\mathrm{d}r}^2\big). \tag{6.24}$$

Consider $(1-t)\,r + t\,s \circ \Phi|_P : P \to \mathbb{C}$ for $t \in [0,1]$. The derivative of a function in $I_{\mathrm{d}r}^2$ lies in $I_{\mathrm{d}r} \cdot I_{\mathrm{d}r,\partial^2 r}$, where $I_{\mathrm{d}r,\partial^2 r}$ is the ideal generated by the first and second derivatives of r. Thus, differentiating (6.24) shows that

$$\mathrm{d}\big((1-t)r + ts \circ \Phi|_P\big) = \big[\mathrm{id}_{T^*P} + t\,\alpha\big] \circ \mathrm{d}r \quad \text{for } \alpha \in I_{\mathrm{d}r,\partial^2 r}|_P \cdot \operatorname{End}(T^*P).$$

We have $\mathrm{d}r|_{j(x)} = \partial^2 r|_{j(x)} = 0$ as $j(x) \in \operatorname{Crit}(r)$ and $T_{j(x)}P = T_{j(x)} \operatorname{Crit}(r)$. So $\alpha|_{j(x)} = 0$, and α is small near $j(x)$. Making P smaller, we can suppose that $\mathrm{id}_{T^*P} + t\,\alpha : T^*P \to T^*P$ is invertible for all $t \in [0,1]$. So $I_{\mathrm{d}r}$ is also generated by $\mathrm{d}\big((1-t)r + ts \circ \Phi|_P\big)$ for each $t \in [0,1]$.

We now copy the proof of Proposition 4.3, to construct open neighbourhoods P', P'' of $j(x)$ in P and a biholomorphism $\Theta : P' \to P''$ such that $\Theta|_{\operatorname{Crit}(r) \cap P'} = \mathrm{id}_{\operatorname{Crit}(r) \cap P'}$ and $s \circ \Phi \circ \Theta = r|_{P'}$. There is a difference: as we work modulo the square $I_{\mathrm{d}r}^2$ of $I_{\mathrm{d}r}$, rather than the cube I_X^3 of I_X as in Proposition 4.3, we cannot also prove that $\mathrm{d}\Theta|_{TP|_{\operatorname{Crit}(r) \cap P'}} = \mathrm{id}_{TP|_{\operatorname{Crit}(r) \cap P'}}$. The important point to make the proof work modulo $I_{\mathrm{d}r}^2$ rather than modulo I_X^3 is that $I_{\mathrm{d}r}$ is generated by $\mathrm{d}\big((1-t)r + ts \circ \Phi|_P\big)$ for each $t \in [0,1]$.

From above, $\operatorname{Crit}(r|_{P'})$ is an open neighbourhood of $j(x)$ in $\operatorname{Crit}(f)$. Write $Q' = \Phi \circ \Theta(P') = \Phi(P'')$, so that Q' is an open neighbourhood of $k(x)$ in Q, and $\Phi \circ \Theta : P' \to Q'$ is a biholomorphism. As $s \circ \Phi \circ \Theta = r|_{P'}$, we see that $\Phi \circ \Theta$ induces an isomorphism $\operatorname{Crit}(r|_{P'}) \to \operatorname{Crit}(r|_{Q'})$. But Θ is the identity on $\operatorname{Crit}(r) = \operatorname{Crit}(f)$ locally, and Φ is an isomorphism $\operatorname{Crit}(f) \to \operatorname{Crit}(g)$ locally. Hence $\Phi \circ \Theta$ is a local isomorphism from $\operatorname{Crit}(r)$ to both $\operatorname{Crit}(s|_{Q'})$ and $\operatorname{Crit}(g)$. It follows that $\operatorname{Crit}(s|_{Q'})$ is an open neighbourhood of $k(x)$ in $\operatorname{Crit}(g)$.

Theorem 5.1(i) now shows that there exist an open neighbourhood $\hat V$ of $j(x)$ in V and a holomorphic map $\Lambda : \hat V \to P' \times \mathbb{C}^m$, which is a biholomorphism with an open neighbourhood of $(j(x), 0)$ in $P' \times \mathbb{C}^m$, where $m = \dim V - \dim P$, such that if $p \in P' \cap \hat V$ then $\Lambda(p) = \big(p, (0, \ldots, 0)\big)$, and if $v \in \hat V$ with $\Lambda(v) = \big(p, (y_1, \ldots, y_m)\big)$, then $f(v) = r(p) + y_1^2 + \cdots + y_m^2$.

Similarly, there exist open $k(x) \in \hat{W} \subseteq W$ and holomorphic $\Xi : \hat{W} \to Q' \times \mathbb{C}^{m+n}$, which is a biholomorphism with an open neighbourhood of $(k(x), 0)$, such that if $q \in Q' \cap \hat{W}$ then $\Xi(q) = \bigl(q, (0, \ldots, 0)\bigr)$, and if $w \in \hat{W}$ with $\Xi(w) = \bigl(q, (y_1, \ldots, y_m, z_1, \ldots, z_n)\bigr)$, then $g(w) = s(q) + y_1^2 + \cdots + y_m^2 + z_1^2 + \cdots + z_n^2$.

Now define open $k(x) \in W' \subseteq W$ and holomorphic $\Psi : W' \to V \times \mathbb{C}^n$ by

$$
\begin{aligned}
W' &= \Xi^{-1}\bigl(((\Phi \circ \Theta) \times \mathrm{id}_{\mathbb{C}^{m+n}})(\Lambda(\hat{V}) \times \mathbb{C}^n)\bigr), \\
\Psi &= (\Lambda \times \mathrm{id}_{\mathbb{C}^n})^{-1} \circ ((\Phi \circ \Theta) \times \mathrm{id}_{\mathbb{C}^{m+n}})^{-1} \circ \Xi.
\end{aligned}
\tag{6.25}
$$

Since Θ, Λ, Ξ are biholomorphisms with their images, Ψ is a biholomorphism with its image. Setting $X' = k^{-1}(W')$, we have

$$
\begin{aligned}
\Phi|_{P''} \circ j|_{X'} &= k|_{X'} : X' \longrightarrow Q', & \Theta \circ j|_{X'} &= k|_{X'} : X' \longrightarrow P'', \\
\Lambda \circ j|_{X'} &= j|_{X'} \times 0 : X' \to P' \times \mathbb{C}^m, & \Xi \circ k|_{X'} &= k|_{X'} \times 0 : X' \to Q' \times \mathbb{C}^{m+n}.
\end{aligned}
$$

Combining these with (6.25) gives $\Psi \circ k|_{X'} = (j \times 0)|_{X'}$. Also, if $w \in W'$ with $\Xi(w) = \bigl(q, (y_1, \ldots, y_m, z_1, \ldots, z_n)\bigr)$ for $q \in Q'$, and $q = \Phi \circ \Theta(p)$ for $p \in P'$, and $\Lambda(v) = \bigl(p, (y_1, \ldots, y_m)\bigr)$ for $v \in \hat{V}$, then $\Psi(w) = \bigl(v, (z_1, \ldots, z_n)\bigr)$, and

$$
\begin{aligned}
g(w) &= s(q) + y_1^2 + \cdots + y_m^2 + z_1^2 + \cdots + z_n^2 \\
&= r(p) + y_1^2 + \cdots + y_m^2 + z_1^2 + \cdots + z_n^2 = f(v) + z_1^2 + \cdots + z_n^2.
\end{aligned}
$$

This proves the first part of Theorem 6.4(b).

For the second part, suppose such W', Ψ exist for each $x \in X$. Given such x, W', Ψ, choose an open neighbourhood V' of $j(x)$ in V such that $V' \times \{0\} \subseteq \Psi(W')$, and define $\Phi : V' \to W$ by $\Phi(v) = \Psi^{-1}\bigl(v, (0, \ldots, 0)\bigr)$ for $v \in V'$, and $X' = j^{-1}(V')$. Then $\Phi \circ j|_{X'} = k|_{X'}$, as $\Psi \circ k|_{X'} = (j \times 0)|_{X'}$, and $f|_{V'} = g \circ \Phi$, so (6.8) holds. Thus, for each $x \in X$ we can choose V', Φ as in Theorem 6.4(a) so that (6.8) holds. As this condition is independent of the choice of V', Φ near x, it holds for all choices of V', Φ, so $(V, f, j), (W, g, k)$ are compatible.

6.4 Theorem 6.4(c): relation with Theorem 6.1

Let $(V, f, j), (W, g, k)$ and x, W', Ψ be as in part (b). Then we have a commutative diagram of analytic coherent sheaves on X, with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & TX \arrow[r] \arrow[d, "\mathrm{id}"', "\cong"] & {\begin{array}{c} j^*(TV) \\ \oplus T_0\mathbb{C}^n \end{array}} \arrow[rr, "\begin{pmatrix} j^*(\mathrm{Hess}\, f) & 0 \\ 0 & \mathrm{d}z_1^2 + \cdots + \mathrm{d}z_n^2 \end{pmatrix}"] \arrow[d, "k^*(\mathrm{d}\Psi^{-1})"', "\cong"] & & {\begin{array}{c} j^*(T^*V) \\ \oplus T_0^*\mathbb{C}^n \end{array}} \arrow[r] \arrow[d, "\cong"', "k^*(\mathrm{d}\Psi^*)"] & T^*X \arrow[r] \arrow[d, "\cong"', "\mathrm{id}"] & 0 \\
0 \arrow[r] & TX \arrow[r] & k^*(TW) \arrow[rr, "k^*(\mathrm{Hess}\, g)"] & & k^*(T^*W) \arrow[r] & T^*X \arrow[r] & 0.
\end{tikzcd}
\tag{6.26}
$$

As in Theorem 6.1(i), the exact sequences (6.1) induce isomorphisms of line bundles α, β in (6.2). But (6.1) appear as the two rows of (6.26), where the top

row is summed with $0 \to T_0\mathbb{C}^n \to T_0^*\mathbb{C}^n \to 0$. So by properties of determinant line bundles, we get a commutative diagram

$$\begin{tikzcd}[column sep=huge] \begin{array}{c}\big(\det(TX)\otimes j^*(K_V)\big)\\ \otimes\Lambda^n T_0^*\mathbb{C}^n\end{array} \arrow[r, "\alpha\otimes(\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n)^{-2}"'] \arrow[d, "\Lambda^{\dim W}(k|_{X'}^*(\mathrm{d}\Psi^*))\otimes\mathrm{id}_{\det(TX)}"] & \begin{array}{c}\big(j^*(K_V)^*\otimes\det(T^*X)\big)\\ \otimes\Lambda^n T_0\mathbb{C}^n\end{array} \arrow[d, "\Lambda^{\dim W}(k|_{X'}^*(\mathrm{d}\Psi^{-1}))\otimes\mathrm{id}_{\det(T^*X)}"'] \\ \det(TX)\otimes k^*(K_W) \arrow[r, "\beta"] & k^*(K_W)^*\otimes\det(T^*X) \end{tikzcd} \tag{6.27}$$

of isomorphisms of line bundles on X.

Now (6.3) says that up to canonical isomorphisms of line bundles, we have

$$\Upsilon_{f,g} \cong \alpha\otimes\beta^{-1}. \tag{6.28}$$

Similarly, equations (6.9) and (6.27) say that

$$\iota_s \cong \Lambda^{\dim W}(k|_{X'}^*(\mathrm{d}\Psi^*))\otimes(\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n), \tag{6.29}$$

$$\alpha\otimes\beta^{-1} \cong \big(\Lambda^{\dim W}(k|_{X'}^*(\mathrm{d}\Psi^*))\otimes(\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n)\big)^2. \tag{6.30}$$

Combining (6.28)–(6.30) gives $\iota_s\otimes\iota_s = \Upsilon_{f,g}$, which is equation (6.10). This completes the proof of Theorem 6.4.

6.5 Theorem 6.7(i): canonical isomorphisms $\Delta_{f,g}$

Let (V,f,j) and (W,g,k) be compatible triples with $\dim V \leqslant \dim W$, and write $n = \dim W - \dim V$. For each $x\in X$, choose $x\in W'_x\subseteq W$ and $\Psi_x : W'_x \to V\times\mathbb{C}^n$ with $g|_{W'_x} = (f\boxplus z_1^2+\cdots+z_n^2)\circ\Psi_x$ as in Theorem 6.4(b), where we now use 'x' as a subscript to distinguish choices for different $x\in X$. Write $X'_x = k^{-1}(W'_x)$, so that X'_x is an open neighbourhood of x in X, and $\Psi_x\circ k|_{X'_x} = (j\times 0)|_{X'_x}$. Write $\iota_{s_x} : j^*(K_V)|_{X'_x} \to k^*(K_W)|_{X'_x}$ for the isomorphism defined in (6.9), and s_x for the corresponding section of $Q_{f,g}|_{X'_x}\to X'_x$ defined in Theorem 6.4(c).

Define an isomorphism $\Delta_x : j^*(\mathcal{PV}^\bullet_{V,f})|_{X'_x} \to k^*(\mathcal{PV}^\bullet_{W,g})\otimes_{\mathbb{Z}_2} Q_{f,g}|_{X'_x}$ in $\mathrm{Perv}(X'_x)$ as for (5.10), by the commutative diagram:

$$\begin{tikzcd} j^*(\mathcal{PV}^\bullet_{V,f})|_{X'_x} \arrow[r, "\alpha_x"'] \arrow[dd, "\Delta_x"] & j^*(\mathcal{PV}^\bullet_{V,f})|_{X'_x} \overset{L}{\boxtimes} \mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \arrow[d, "\beta_x"'] \\ & (j\times 0)^*\big(\mathcal{PV}^\bullet_{V\times\mathbb{C}^n,f\boxplus z_1^2+\cdots+z_n^2}\big)\big|_{X'_x} \arrow[d, "\gamma_x"'] \\ k^*(\mathcal{PV}^\bullet_{W,g})\otimes_{\mathbb{Z}_2} Q_{f,g}\big|_{X'_x} & k^*(\mathcal{PV}^\bullet_{W,g})|_{X'_x}. \arrow[l, "\delta_x"'] \end{tikzcd} \tag{6.31}$$

Here the isomorphisms $\alpha_x,\ldots,\delta_x$ in (6.31) are defined as follows:

(i) α_x comes from the isomorphism $\mathcal{PV}^\bullet_{\mathbb{C}^n,z_1^2+\cdots+z_n^2} \cong \mathbb{Q}_{\{0\}}$, which was constructed in Example 2.16.

(ii) β_x comes from the Thom–Sebastiani Theorem for $\mathcal{PV}^\bullet_{V,f}$, Theorem 2.15.

(iii) γ_x is defined in a similar way to Φ_* in (2.10) for $\Psi_x : W'_x \to V \times \mathbb{C}^n$.

(iv) δ_x is induced by the section s_x of $Q_{f,g}|_{X'_x} \to X'_x$.

Now let $x, y \in X$. The proofs of (5.11) and (5.12) yield

$$\begin{aligned} \alpha_x|_{X'_x\cap X'_y} = \alpha_y|_{X'_x\cap X'_y}, \quad \beta_x|_{X'_x\cap X'_y} = \beta_y|_{X'_x\cap X'_y}, \quad \text{and} \\ \gamma_x|_{X'_x\cap X'_y} = \det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr)\cdot\gamma_y|_{X'_x\cap X'_y}, \end{aligned} \tag{6.32}$$

where $\det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr) : X'_x \cap X'_y \to \{\pm 1\}$.

In the restrictions to $X'_x \cap X'_y$ of (6.9) for x, W'_x, Ψ_x and y, W'_y, Ψ_y defining ι_{s_x}, ι_{s_y}, the morphisms $\Lambda^{\dim W}(k|^*_{X'_x\cap X'_y}(\mathrm{d}\Psi_x^*))$ and $\Lambda^{\dim W}(k|^*_{X'_x\cap X'_y}(\mathrm{d}\Psi_y^*))$ differ by a factor $\det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr)$. Thus

$$\iota_{s_y}|_{X'_x\cap X'_y} = \det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr)\cdot\iota_{s_x}|_{X'_x\cap X'_y}.$$

So s_x, s_y also differ by $\det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr)$ on $X'_x \cap X'_y$, and therefore as for (5.13) we have

$$\delta_y|_{X'_x\cap X'_y} = \det\bigl(\mathrm{d}(\Psi_y^{-1}\circ\Psi_x)|_{X'_x\cap X'_y}\bigr)\cdot\delta_x|_{X'_x\cap X'_y}. \tag{6.33}$$

Combining (6.32)–(6.33), we see that

$$\Delta_x|_{X'_x\cap X'_y} = (\delta_x\circ\gamma_x\circ\beta_x\circ\alpha_x)|_{X'_x\cap X'_y} = (\delta_y\circ\gamma_y\circ\beta_y\circ\alpha_y)|_{X'_x\cap X'_y} = \Delta_y|_{X'_x\cap X'_y}.$$

We have chosen an open cover $\{X'_x : x \in X\}$ for X, and on each X'_x we have defined an isomorphism $\Delta_x : j^*(\mathcal{PV}^\bullet_{V,f})|_{X'_x} \to k^*(\mathcal{PV}^\bullet_{W,g}) \otimes_{\mathbb{Z}_2} Q_{f,g}|_{X'_x}$, such that on overlaps $X'_x \cap X'_y$ we have $\Delta_x|_{X'_x\cap X'_y} = \Delta_y|_{X'_x\cap X'_y}$. Therefore by Theorem 2.5(i), there exists a unique isomorphism $\Delta_{f,g}$ as in (6.12) such that $\Delta_{f,g}|_{X'_x} = \Delta_x$ for all $x \in X$. The argument in §5.3 for $\Theta_{f,g}$ shows that $\Delta_{f,g}$ is independent of the choices of W'_x, Ψ_x. We prove that (6.13) commutes as for the proof of (5.3) in §5.3. This completes Theorem 6.7(i) when $\dim V \leqslant \dim W$.

If $\dim V > \dim W$, then exchanging $(V, f, j), (W, g, k)$ above gives an isomorphism $\Delta_{g,f} : k^*(\mathcal{PV}^\bullet_{W,g}) \to j^*(\mathcal{PV}^\bullet_{V,f}) \otimes_{\mathbb{Z}_2} Q_{g,f}$. Note too that in Theorem 6.1 we have $\Upsilon_{g,f} = \Upsilon_{f,g}^{-1}$ by (6.4), so the principal $\mathbb{Z}_2$-bundles $Q_{g,f}, Q_{f,g}$ are inverse (equivalently, isomorphic), and thus there is a natural isomorphism $\iota : \mathbb{Z}_2 \times X \to Q_{g,f} \otimes_{\mathbb{Z}_2} Q_{f,g}$, with $\mathbb{Z}_2 \times X \to X$ the trivial principal $\mathbb{Z}_2$-bundle. Define $\Delta_{f,g}$ in (6.12) to be the unique isomorphism in the commutative diagram

$$\begin{tikzcd} j^*(\mathcal{PV}^\bullet_{V,f}) \arrow[rr, "\Delta_{f,g}"'] \arrow[d, equal] & & k^*(\mathcal{PV}^\bullet_{W,g}) \otimes_{\mathbb{Z}_2} Q_{f,g} \arrow[d, "\Delta_{g,f}\otimes\mathrm{id}_{Q_{f,g}}"'] \\ j^*(\mathcal{PV}^\bullet_{V,f}) \otimes_{\mathbb{Z}_2} (\mathbb{Z}_2 \times X) \arrow[r, "\mathrm{id}_{j^*(\mathcal{PV}^\bullet_{V,f})}\otimes\iota"] & j^*(\mathcal{PV}^\bullet_{V,f}) \otimes_{\mathbb{Z}_2} Q_{g,f} \otimes_{\mathbb{Z}_2} Q_{f,g}. \end{tikzcd} \tag{6.34}$$

Essentially this just says that $\Delta_{f,g} = \Delta_{g,f}^{-1}$, modulo canonical isomorphisms of $\mathbb{Z}_2$-bundles. We can deduce that (6.13) commutes for $\Delta_{f,g}$ from the fact that (6.13) commutes for $\Delta_{g,f}$. This completes the proof of Theorem 6.7(i).

6.6 Theorem 6.7(ii): composing $\Delta_{e,f}, \Delta_{f,g}$

First suppose $(U, e, i), (V, f, j), (W, g, k)$ are compatible triples with $\dim U \leqslant \dim V \leqslant \dim W$. We can prove (6.14) commutes by a very similar argument to that in §5.4 showing that (5.4) commutes. The proof of the analogue of (5.23) is essentially unchanged; for the analogue of (5.24), we must check that the sections s_x used to define δ_x in part (iv) of §6.5, in the definitions of each of $\Delta_{e,f}, \Delta_{f,g}, \Delta_{e,g}$, are compatible with the isomorphism $\Gamma_{e,f,g}$ in Theorem 6.1(ii).

If we do not have $\dim U \leqslant \dim V \leqslant \dim W$, then we can deduce (6.14) commutes by combining (6.14) for a permutation of $(U, e, i), (V, f, j), (W, g, k)$ with increasing dimensions, with equation (6.34). For example, suppose $\dim V < \dim U \leqslant \dim W$, and consider the diagram:

$$
\begin{tikzcd}
i^*(\mathcal{PV}^\bullet_{U,e}) \arrow[rr, "\Delta_{e,f}"'] \arrow[dr, "\mathrm{id}_{i^*(\mathcal{PV}^\bullet_{U,e})}\otimes\iota"'] \arrow[ddd, "\Delta_{e,g}"] & & j^*(\mathcal{PV}^\bullet_{V,f}) \otimes_{\mathbb{Z}_2} Q_{e,f} \arrow[dl, "\Delta_{f,e}\otimes\mathrm{id}_{Q_{e,f}}"] \arrow[ddd, "\Delta_{f,g}\otimes\mathrm{id}_{Q_{e,f}}"'] \\
& \begin{array}{c} i^*(\mathcal{PV}^\bullet_{U,e})\otimes_{\mathbb{Z}_2} \\ Q_{f,e}\otimes_{\mathbb{Z}_2} Q_{e,f} \end{array} \arrow[d, "\Delta_{e,g}\otimes\mathrm{id}_{Q_{f,e}\otimes Q_{e,f}}"] & \\
& \begin{array}{c} k^*(\mathcal{PV}^\bullet_{W,g}) \otimes_{\mathbb{Z}_2} Q_{e,g} \\ \otimes_{\mathbb{Z}_2} Q_{f,e} \otimes_{\mathbb{Z}_2} Q_{e,f} \end{array} \arrow[dl, "\mathrm{id}_{k^*(\mathcal{PV}^\bullet_{W,g})\otimes Q_{e,g}}\otimes\iota^{-1}"'] \arrow[dr, shift right, "\mathrm{id}\otimes\Gamma^{-1}_{f,e,g}\otimes\mathrm{id}_{Q_{e,f}}"'] & \\
k^*(\mathcal{PV}^\bullet_{W,g})\otimes_{\mathbb{Z}_2} Q_{e,g} \arrow[rr, "\mathrm{id}\otimes\Gamma_{e,f,g}"] & & k^*(\mathcal{PV}^\bullet_{W,g})\otimes_{\mathbb{Z}_2} Q_{f,g}\otimes_{\mathbb{Z}_2} Q_{e,f}. \arrow[ul, shift right, "\mathrm{id}\otimes\Gamma_{f,e,g}\otimes\mathrm{id}_{Q_{e,f}}"']
\end{tikzcd}
\tag{6.35}
$$

Here the upper triangle commutes by (6.34) for $(U, e, i), (V, f, j)$, the left hand quadrilateral commutes as $\iota^{-1} \circ \iota = \mathrm{id}$, the right hand quadrilateral commutes by (6.14) with $(U, e, i), (V, f, j)$ exchanged, as above, and the lower triangle commutes by the definitions of $\Gamma_{e,f,g}, \Gamma_{f,e,g}, \iota$. So (6.35) commutes. The outer rectangle shows that (6.14) commutes when $\dim V < \dim U \leqslant \dim W$.

6.7 Theorem 6.7(iii): extension to mixed Hodge modules

As in §3–§5, this extension is now routine: the diagram (6.31) can be lifted to Hodge modules, compatibility with monodromy ensures that the isomorphism is one of mixed Hodge modules with monodromy, and the sheaf property and faithfulness ensure the necessary compatibilities.

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Address for all five authors:
The Mathematical Institute, 24-29 St. Giles, Oxford, OX1 3LB, U.K.
E-mails: brav@maths.ox.ac.uk, bussi@maths.ox.ac.uk,
dupont@maths.ox.ac.uk, joyce@maths.ox.ac.uk, szendroi@maths.ox.ac.uk.